Ain Shams University
Faculty of Science
Department of Mathematics

On Extensions of Injectivity of Graded Modules

A Thesis

submitted in Partial Fulfillment for
the requirements of the Degree of Master of Science
(Pure Mathematics)

By
Hesham Sherif El-Sayed Hussein
Department of Mathematics
Faculty of Science, Ain shams University

Supervised By

Prof. Dr. Salah El Din Sayed Hussein
Professor of Pure Mathematics
Department of Mathematics
Faculty of Science
Ain Shams University

Dr. Essam Ahmed Soliman El-Seidy
Assistant Professor of Pure Mathematics
Department of Mathematics
Faculty of Science
Ain Shams University

Dr. Samir Sayed Mahmoud Mohamed
Lecturer of Pure Mathematics
Department of Mathematics
Faculty of Science
Ain Shams University

To

Department of Mathematics
Faculty of Science
Ain Shams University
Cairo, Egypt
2015

ACKNOWLEDGEMENT

I humbly acknowledge the blessings of **Almighty**, **Compeller and Subdeur Allah**, without **Whose**mercy and guidance this work neverhas been started nor completed. May **Allah**prays on Mohamed "peaceBe Upon Him" the prophet and the Messenger of **Allah**.

I would like to express my deep gratitude to **Prof.Dr. Salah El-Din SayedHussien , Dr. Essam Ahmed Soliman El- Seidy, and Dr.SamirSayed Mahmoud Mohamed**, Department of Mathematics ,Faculty of Science, Ain Shams University for their patience, criticism, advice and much help. I am grateful for their careful review of the manuscript.
I do not forget their encouragement, advice in reading the background material for this work and for giving so generously of their time to completethis thesis in final form.

I wish to record my gratitude to the staff of Department of Mathematics,Faculty of Science, Ain Shams University, for facilities extended to me and giving me opportunity of further my education.

Many thanks also go to all my colleagues, in my department whom
wished for me a good luck, wishing for them all the best in the life.

To my family, I offer my sincere gratitude for their love and support and
giving me opportunity of further my education.

Contents

Abstract

This thesis has two predominant objectives. On one hand, we investigate the relation between the concept of graded essential extensions and graded injectivity of graded modules. Actually, a characterization of the graded injective hull in terms of graded essential extensions is deduced and graded versions of (Echman-Schopf) theorem and (Papp-Bass) theorem has been proved. On the other hand, we study one of the most important extensions of graded injectivity, namely the graded soc-injectivity. Roughly speaking, after defining the graded soc-injectivity of graded modules we provide characterizations of semiartinian graded rings, graded noetherian rings, and graded quasi-Frobenius rings in terms of graded soc-injectivity.

Summary

It is undoubtedly that modules and rings are ubiquitous objects of abstract algebra and they are delight to construct. A natural strategy for investigating the structure of a ring is to consider a given set of conditions on the modules it admits. The study of rings and modules would be deficient without a connection with homological algebra. In the solar system of homological algebra the sun is certainly the theory of projective and injective modules. If R is an associative ring with 1 then every R-module M can be embedded in an injective R-module. Among the injective modules that contain M there is a minimal one called the injective hull of M, denoted by $E(M)$. The injective hull of M is unique up to isomorphism and in some sense, it is the best approximation of M by an injective module. The concept of essential extensions will prove to be indispensable in dealing with uniform dimensions and in the formation of the injective hull of modules which is crucial for the theory of rings of quotients. Injective modules are closely related to essential extensions. Actually, a module is injective if and only if it has no proper essential extensions. During the past twenty years, the theory of injective modules has enjoyed a period of

vigorous development. The concept of injectivity has been extended and characterizations of quasi-Frobenius rings, perfect rings, Kasch rings, and semiartinian rings in terms of extended injectivities of modules have been established,[1],[9],[15],[20].

The motivation underlying graded rings and modules is almost contrary to the one of representation theory. The graded methods are not aiming to obtain information about the grading group of the graded ring $R = \oplus_{\sigma \in G} R_\sigma$. On the contrary, the existence of a G-gradation is used to relate R and R_e, e is the unit element of G, or graded to ungraded properties of R. Graded rings have been proved to be a very satisfactory tool in the study of algebraic geometry where they are used to gain information about projective varieties, [21],[10].

This thesis has two predominant objectives. On one hand, we investigate the relation between the concept of graded essential extensions and graded injectivity of graded modules. Actually, a characterization of the graded injective hull in terms of graded essential extensions is deduced and graded versions of (Echman-Schopf) theorem [16] and (Papp-Bass) theorem [4] has been proved. On the other hand, we study one of the most important extensions of graded injectivity, namely the graded soc-injectivity. Roughly speaking, after defining the graded soc-injectivity of graded modules we provide characterizations of semiartinian graded rings, graded noetherian rings, and graded quasi-Frobenius rings in terms of graded soc-injectivity. The new results we have obtained so far are displayed in the third chapter of this dissertation.

The thesis consists of three chapters.

The first chapter provides the preliminaries and some background material to be used in the subsequent chapters. The second chapter consists of two sections.

The first section is devoted to the study of graded injective module. We proved Baer's theorem for injectivity of graded modules. Actually, the proof is a slight modification of the proof in the ungraded case. The importance of Baer's criterion is a clear consequence of its iterated use throughout the thesis. The relation between injectivity and graded injectivity is discussed. In the second section we investigate the concept of H-injective graded modules, where H is a subgroup of the grading group G, as displayed in [18]. The well-known result of Higman about H-injective modules over strongly graded rings is presented.

The third chapter is divided into two sections.

In the first section we study basic properties of essential extensions of graded modules. The graded injective hull of a graded module is characterized in terms of graded essential extensions. The graded versions of the well-known Echman-Schopf theorem and Papp-Bass theorem for injective modules over noetherian rings have been proved.

The second section is devoted to the study of one of the most important extended injectivity concepts of graded modules which is the graded soc-injectivity. Actually, we introduce and investigate the concepts of graded soc-injectivity and strongly soc-injectivity. We study the relation between these concepts and the graded semi-simplicity as well as the graded projectivity of graded modules. We round off by establishing important characterization of graded semiartinian rings, graded quasi-Frobenius rings, and graded noetherian rings in terms of graded soc-injectivity of graded modules.

The main results of this thesis are published in

Pure and Apllied Mathematics Journal, Science Publishing Group (SPG), 2015, Vol. 4, No. 2, 47-51.

Chapter 1

Preliminaries

In this chapter we assemble the most basic known concepts and their associated necessary results for our study in this thesis. For details on injectivity and projectivity of modules we refer to [2],[3],[7],[8] and [22] while for details on graded rings and graded modules we refer to [10],[11], and [14].

1.1 Projective and Injective Modules

Throughout R is an associative ring with 1. All R-modules are left R-modules unless otherwise stated. By R-mod we denote the category of left R-modules.

Definition 1.1.1 *,[6]. Let M be an R-module. A set of elements X in M is said to generate M if for any element $m \in M$, $m = \lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n$ for some $\lambda_1, ..., \lambda_n \in R$, $x_1, ..., x_n \in X$. M is said to be finitely generated if there exist a finite subset X of M which generates M. The elements of such a set are called generators of M.*

Definition 1.1.2 *,[6]. Let M be an R-module. A set X in M is called a basis for M if*

1. *X generates M.*

2. *If $\sum_{i=1}^{n} \lambda_i x_i = 0$, then $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$ for $\lambda_i \in R, x_i \in X$. In other words, X is linearly independent set of elements.*

Example 1.1.3 *,[7]. If $\mathbb{Q}$ is the set of rationals, then the $\mathbb{Z}$-module $\mathbb{Q}$ is not finitely generated.*

proof: We first claim that: if finitely many arbitrary elements are omitted from an arbitrary generating set X of ${}_{\mathbb{Z}}\mathbb{Q}$, then the set with these elements omitted is again a generating set of ${}_{\mathbb{Z}}\mathbb{Q}$.

Indeed, let $x_o \in X$, since X is a generating set for $\mathbb{Q}$ then

$$\tfrac{x_o}{2} = z_o x_o + \textstyle\sum_{x_i \neq x_o} z_i x_i \qquad , z_i \in \mathbb{Z}, x_i \in X$$

then it follows that

$$x_o = 2z_o x_o + \sum_{x_i \neq x_o} 2z_i x_i \Rightarrow n x_o = \sum_{x_i \neq x_o} 2z_i x_i$$

where $n = 1 - 2z_o \in \mathbb{Z}$ and $n \neq 0$. Now let

$$\tfrac{x_o}{n} = z_o' x_o + \textstyle\sum_{x_i \neq x_o} z_i' x_i \qquad , z_i' \in \mathbb{Z}, x_i \in X$$

hence

$$\begin{aligned} x_o &= nz_o^{'}x_o + \sum_{x_i \neq x_o} nz_i^{'}x_i \\ &= \sum_{x_i \neq x_o} 2z_o^{'}z_i x_i + \sum_{x_i \neq x_o} nz_i^{'}x_i \\ &= \sum_{x_k \neq x_o} z_k^{'}x_k \qquad , x_k \in X, z_k \in \mathbb{Z} \end{aligned}$$

Thus x_o *lies in the submodule generated by* $X - \{x_o\}$*, and since* X *is a generating set of* ${}_{\mathbb{Z}}\mathbb{Q}$*, then so also is* $X - \{x_o\}$*. The assertion follows by induction. Hence there is no finite set of generators of* ${}_{\mathbb{Z}}\mathbb{Q}$*, since otherwise* ${}_{\mathbb{Z}}\mathbb{Q}$ *would be generated by the empty set and it would follow that* ${}_{\mathbb{Z}}\mathbb{Q} = 0$*.*

Definition 1.1.4 *,[6]. An R-module M is said to be free if it satisfies one (and hence both) of the following equivalent conditions:*

1. $M \cong \oplus_{i \in I} R_i$ *such that* $R_i = R$*, for every* $i \in I$*.*

2. M *admits a basis.*

Lemma 1.1.5 *,[7]. Every R-module M is an epimorphic image of a free R-module. If M is finitely generated, then M is an epimorphic image of a free R-module with a finite basis.*

Example 1.1.6 *1. Any finitely generated vector space* V *over a field* F *is a free* F*-module of finite basis.*

2. $\mathbb{Z}_2$ *as* $\mathbb{Z}$*-module is not free, since* $\mathbb{Z}_2$ *is finitely generated by* $\overline{1}$ *but* $\{\overline{1}\}$ *is not linearly independent (obviously* $2.\overline{1} = \overline{0}$*).*

3. *If the ring R is viewed as an R-module, then R is a free R-module with basis* $\{1\}$.

4. *The R-module* $\mathbb{R}^n$ *is a free R-module with basis* $\{e_i\}_{i=1}^{n}$, *where*

$$e_1 = (1, 0, 0, ..., 0),$$

$$e_2 = (0, 1, 0, ..., 0),$$

$$e_3 = (0, 0, 1, ..., 0),$$

and ...

$$e_n = (0, 0, 0, ..., 1)$$

5. *The matrix ring* $M_n(\mathbb{R})$ *is a free R-module. One basis of* $M_n(\mathbb{R})$ *is the set of matrix units* $\{E_{ij}\}_{i,j=1}^{n}$ *with* n^2 *elements. where*

$$E_{i,j} = \begin{pmatrix} e_{l,k} \end{pmatrix}_{l,k=1}^{n}$$

is given by $e_{l,k} = \begin{cases} 1, & \text{l=i and k=j} \\ 0, & \text{otherwise} \end{cases}$

For example, if $(a_{ij}) \in M_2(\mathbb{R})$, *then*

$$(a_{ij}) = a_{11}E_{11} + a_{12}E_{12} + a_{21}E_{21} + a_{22}E_{22}$$

Theorem 1.1.7 *,[10]. Let P be an R-module. The following statements are equivalent:*

1. *P is a direct summand of a free R-module.*

2. *Given any diagram*

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow h & & \\ A & \xrightarrow{f} & B & \longrightarrow & 0 \end{array}$$

with exact bottom row, there exists an R-homomorphism $g : P \to A$ *such that* $fg = h$.

3. *Every exact sequence* $0 \to M' \to M \to P \to 0$ *splits.*

Definition 1.1.8 *An R-module P satisfying one (hence all) of the above conditions is called a projective left R-module.*

Proposition 1.1.9 *,[7]. If* $f : N \to M$ *is an epimorphism and M is a projective R-module, then M is isomorphic to a direct summand of N.*

Example 1.1.10

1. *Every vector space over a division ring is projective.*
2. *Every free R-module is projective.*
3. *There are modules that are not projective. For example,* $\prod_N \mathbb{Z}_i$, *where* $\mathbb{Z} = \mathbb{Z}_i$ *for* $i = 1, 2, 3, \ldots$ *, is not a projective* $\mathbb{Z}$*-module,[8].*
4. *The* $n \times n$ *matrix ring* $M_n(\mathbb{R})$ *is projective as an* $M_n(\mathbb{R})$ *module and as an R-module,[3].*
5. *For a non-zero idempotent e of R, eR is a projective R-module but not free.Indeed, since* $R = eR \oplus (1 - e)R$ *and* ${}_RR$ *is a free R-module then*

eR is a projective R-module. And since

$$(1-e)er = er - e^2r = 0\ ,\ (1-e) \neq 0.$$

Therefore $\{er\}$ *is not a basis for* eR *for all* $r \in R$*. Thus* eR *is not free.*

6. *Since* $\mathbb{Z}_6 = <\overline{2}> \oplus <\overline{3}>$ *then both* $<\overline{2}>$ *and* $<\overline{3}>$ *are projective* $\mathbb{Z}_6$*-modules. On the other hand, non of them is a free* $\mathbb{Z}_6$*-module. Indeed, neither* $<\overline{2}>$ *nor* $<\overline{3}>$ *is isomorphic to direct sum of copies of* $\mathbb{Z}_6$*.*

7. $\mathbb{Q}$ *is* $\mathbb{Z}$*-projective as* $\mathbb{Z}$*-module. Indeed, consider the following diagram*

$$\begin{array}{ccccc} & & \mathbb{Q} & & \\ & & \downarrow f & & \\ \mathbb{Z} & \xrightarrow{\eta} & \mathbb{Z}/n\mathbb{Z} & \xrightarrow{g} & 0 \end{array}$$

where $f : \mathbb{Q} \to \mathbb{Z}/n\mathbb{Z}$ *is defined by*

$$f(q) = f(n\frac{q}{n}) = nf(\frac{q}{n}) = 0 \ (i.e\ f = 0.)$$

then there exists $\theta : \mathbb{Q} \to \mathbb{Z}$ *such that* $\theta = 0$ *with* $\eta\theta = f$*.*

8. $\mathbb{Q}/\mathbb{Z}$ *is* $\mathbb{Z}$*-projective but not* $\mathbb{Q}$*-projective. Indeed, consider the following diagram*

$$\begin{array}{ccccc} & & \mathbb{Q}/\mathbb{Z} & & \\ & & \downarrow f & & \\ \mathbb{Z} & \xrightarrow{\eta} & \mathbb{Z}/n\mathbb{Z} & \xrightarrow{g} & 0 \end{array}$$

where $f : \mathbb{Q}/\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ *is defined by*

$$f(q + Z) = f(n(\frac{q}{n} + Z)) = nf(\frac{q}{n} + Z) = 0 \ (i.e \ f = 0).$$

Obviously there exists $\theta : \mathbb{Q}/\mathbb{Z} \to \mathbb{Z}$ *such that* $\theta = 0$ *with* $\eta\theta = f$. *But if* $\mathbb{Q}/\mathbb{Z}$ *is* $\mathbb{Q}$*-projective then* $\mathbb{Z}$ *is a direct summand of* $\mathbb{Q}$*(Proposition 1.1.9), thus* $\mathbb{Q} = \mathbb{Z} \oplus K$ *for some* $K \leq \mathbb{Q}$. *Since* $\frac{1}{2} \in \mathbb{Q} = \mathbb{Z} \oplus K$ *then* $\frac{1}{2} = z + k$ *for* $z \in \mathbb{Z}, k \in K$. *Thus* $1 = 2z + 2k$ *and* $1 - 2z = 2k \in \mathbb{Z} \cap K = \{0\}$, *therefore* $1 - 2z = 0$. *Hence* $z = \frac{1}{2} \in \mathbb{Z}$, *which is a contradiction.*

Proposition 1.1.11 *,[3]. If* $\{M_\alpha\}_{\alpha\in\Lambda}$ *is a family of R-modules, then* $\oplus_{\alpha\in\Lambda} M_\alpha$. *is projective if and only if each* M_α *is projective.*

Definition 1.1.12 *An R-module Q is said to be injective if and only if it satisfies one (and hence all) of the following equivalent conditions:*

1. *Every short exact sequence*

$$0 \to Q \xrightarrow{f} A \xrightarrow{g} B \to 0$$

splits.