Block Iterative Methods and Acceleration Techniques

Thesis by

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Summary

Thesis title: “Block Iterative Methods and Acceleration Techniques”.

The main objective in this work is based on the fact that the rate of convergence of the block Jacobi method is approximately the same as that of the point Gauss Seidel method. The aim of this thesis is to introduce a new version of group iterative methods for the algebraic systems arising from the discretization of boundary value problems in the plane. Some special forms of a relatively new iterative block technique are introduced, the KSLOR, KS2LOR, and GKSOR.

The thesis consists of four chapters, Arabic summary, and English summary.

Chapter One: Basic Concepts

The basic concepts required in investigating iterative techniques for solving linear systems are introduced in a simple form.

Chapter Two: Block Iterative Methods

Standard Block Iterative methods (Block Jacobi, Block Gauss-Seidel, Block Successive over-relaxation) are studied, moreover the Block KSOR method is introduced. The Block relaxation schemes are generalizations of the point relaxation schemes. They update a whole set of components of unknown vector at each time, typically a sub-vector of the solution vector, instead of only one component. Special block iterative methods are discussed, namely the Line iterative methods (Line Jacobi, Line Gauss-Seidel, Line SOR, and Line KSOR). Also, the two Line iterative methods with the natural ordering as well as the red black ordering are discussed.

Chapter Three: Group Iterative Methods

Different group iterative methods are considered, through the treatment of Poisson equation defined on a unit square with $8 \times 8 = 64$ internal mesh points. The novel approach of using groups of fixed size is

considered. I.e. Groups of a particular number of individual equations (mesh points), each group is dealt in the same way as a single point. Our fundamental concern is to develop new grouping of the mesh points into small size groups of 2, 4, 8 and 16 points and to investigate their advantages as well as choosing the most efficient group for solving such linear systems which arise from the discretization of PDE's in two-space dimensions.

Chapter Four: Preconditioned and Convergence Acceleration

Preconditioning is one of the most usable acceleration techniques. The approach of preconditioning is demonstrated through simple forms of well known types. It is proved that the preconditioning gives considerable improvement in the rate of convergence for the iterative method (Jacobi, Gauss-Seidel, SOR, and KSOR). It is demonstrated that the line SOR method is a more efficient method than preconditioning treatment. Moreover, the 2 line is more efficient than the line method.

- The objective of the thesis is to introduce a new versions of block iterative methods.
- The motivation of the thesis is the Discretization techniques of partial differential equations usually produce matrices which are banded, or block banded.

It is worth to mention that:

- All calculations were done by using Matlab R2012b in the Mathematics Department, Faculty of Science, Ain Shams University.

- The results of chapter two are published in the Journal of "Applied and Computational Mathematics".

Chapter 1:
Basic Concepts

Chapter 1 Basic Concepts

1.1 Introduction

Large systems of algebraic equations appear in the numerical treatment of partial differential equations (PDE's) possess many interesting properties. The matrix formulation of algebraic systems enables the discussion of many difficult concepts and properties such as: irreducibility, diagonal dominance, positive definiteness, and consistently ordered in straightforward approach. These properties have their effects in the iterative treatment of such systems. The basic concepts required in investigating iterative techniques for solving linear systems are introduced.

1.2 Diagonal dominance and irreducibility

Definition 1.2.1: An $(n \times n)$ matrix A is *diagonally dominant* if

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \quad for\ all\ 1 \leq i \leq n, \tag{1.1}$$

and

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \tag{1.2}$$

for at least one i, [26].

Definition 1.2.2: An $(n \times n)$ matrix A is *irreducible* if $n = 1$ or if $n > 1$ and given any two non-empty disjoint subsets S and T of W, the set of the first n positive integers, such that $S \cup T = W$, there exists $i \in S$ and $j \in T$ such that $a_{ij} \neq 0$, [26].

This can be understood from the straightforward definition of reducible systems.

Definition 1.2.3: An $(n \times n)$ matrix A is *reducible* if there exists an $(n \times n)$ permutation matrix P such that

$$P\,A\,P^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \tag{1.3}$$

where A_{11} and A_{22} are submatrices of order $(r \times r)$ and $(n-r) \times (n-r)$, where $1 \leq r < n$. If no such permutation matrix exists, then A is *irreducible*, [26].

The term irreducible was presented by Frobenius 1912; The inspiration for calling matrices *reducible* is quite clear [26], to solve the system $\tilde{A}x = k$, where $\tilde{A} = P\,A\,P^T$. One can partition the vectors x and k similarly so that the system can be written as

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 &= k_1\,, \\ A_{22}x_2 &= k_2\,. \end{aligned} \tag{1.4}$$

Thus, the original system is equivalent to two lower-order subsystems.

The decision of irreducibility is not an easy task; the use of finite directed graph introduced in [26], can be used to simplify this task.

Definition 1.2.4: Let A be an $(n \times n)$ matrix, and consider any n distinct points $P_1, P_2, \ldots, P_n$ in the plane, which we shall call them nodes. For every nonzero element a_{ij} of the matrix A, connect the node P_i to the node P_j by means of an edge $\overrightarrow{P_i P_j}$ directed from P_i to P_j as in Figure 1.1. (For non-zero diagonal elements, a_{ii} the edge goes from P_i to itself forming a loop as in Figure 1.2. The resulting graph is called a *finite directed graph* $G(A)$, [26].

Figure 1.1: path from P_i to P_j

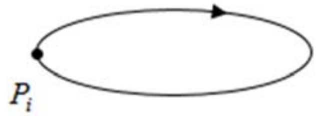

Figure 1.2: loop from P_i to P_i

As an example, consider the two matrices

$$B = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}, \text{and } C = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

$G(B)$ and $G(C)$ are given in figures 1.3 and 1.4, respectively.

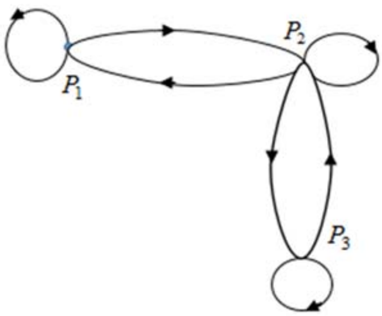

Figure 1.3: directed graph of B

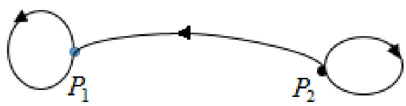

Figure 1.4: directed graph of C

Definition 1.2.5: A directed graph is *strongly connected* if for any ordered pair of nodes P_i and P_j , there exists a directed path

$$\overrightarrow{P_i P_{k_1}}, \overrightarrow{P_{k_1} P_{k_2}}, \dots , \overrightarrow{P_{r-1} P_{r=j}},$$

connecting P_i to P_j. Such a path is said to have length r, [26].

Evidently, the directed graph $G(B)$ in Figure 1.3 is strongly connected. On the other hand $G(C)$ is not strongly connected, since there does not exist a path from P_2 to P_1.

The next theorem describes the relationship between the irreducibility of a matrix and its directed graph.

Theorem 1.2.1: A square matrix A is irreducible if and only if its directed graph $G(A)$ is strongly connected, [26].

Definition 1.2.6: An irreducible matrix, which is also diagonally dominant, with strict inequality holding for at least one i in (1.2) is said to be *irreducibly diagonally dominant,* [26].

If (1.2) holds for all i, then A has *strong diagonal dominance.*

Theorem 1.2.2: If A is irreducibly diagonally dominant matrix, then $\det(A) \neq 0$ and none of the diagonal elements of A vanishes. [31], [26].

A very common matrix in numerical analysis is the $(n \times n)$ tridiagonal matrix

$$A_n = (a_{ij}) = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix},$$

whose directed graph for $n = 6$, $G(A_6)$, is shown in Figure 1.5.

It is clear that $G(A_6)$ is strongly connected, so that A_6 is irreducible, to deduce that A_6 is nonsingular, simply note that $|a_{ii}| \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$, for all $1 \leq i \leq n$, with strict inequality holding for $i = 1$ and $i = n$. Applying Theorem 1.2.2 then gives that A_6 is nonsingular, independent of the size of n.

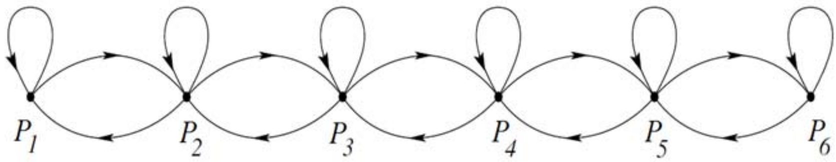

Figure 1.5:$G(A_6)$ for the matrix A_6

1.3 Vector and matrix norms

It is helpful to have some measure of the size or magnitude of a vector or a matrix. This measure is known as a *norm* and is denoted by $\|.\|$.

Definition 1.3.1: The norm of a vector x, denoted by $\|x\|$, is a nonnegative number satisfying the following three axioms

$$\left.\begin{array}{l} 1.\ \|x\| > 0\,,\quad for\ \ x \neq 0\ \ and\ \ \|x\| = 0\ \ iff\ \ x = 0, \\ 2.\ \|\beta x\| = |\beta|\|x\| \quad for\ any\ complex\ scalar\ \beta, \\ 3.\ \|x+y\| \leq \|x\| + \|y\| \quad for\ vectors\ x\ and\ y, \end{array}\right\}, \tag{1.5}$$

also, from (3.), we have

$$\|x-y\| \geq \big|\|x\| - \|y\|\big|. \tag{1.6}$$

The most commonly used norms are L_1 , L_2 and L_∞ , [31].

Definition 1.3.2: If $x = (x_1, x_2, \cdots, x_n)^T$ is an n-dimensional vector then

$$\left.\begin{array}{l} \|x\|_1 = \sum_{i=1}^{n} |x_i|, \\ \|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}, (Euclidean\ norm), \\ \|x\|_\infty = \max_i |x_i|\ \ , (\text{maximum or uniform norm}), \end{array}\right\}. \tag{1.7}$$

In fact, these three norms are special cases of the general L_p norm defined for $p \geq 1$ by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}. \quad (1.8)$$

In a similar fashion, one can proceed to define a matrix norm, [31].

Definition 1.3.3: A norm of an $(n \times n)$ matrix A, written as $\|A\|$, is a scalar satisfying the following axioms

1. $\|A\| > 0 \ \ and \ \|A\| = 0 \ \ iff \ A = 0,$
2. $\|\beta A\| = |\beta|\|A\| \quad for\ any\ scalar\ \beta,$
3. $\|A + B\| \le \|A\| + \|B\| \ \ for\ matrices\ A\ and\ B,$
4. $\|AB\| \le \|A\|\|B\| \quad for\ matrices\ A\ and\ B$, [31], [29].

Definition 1.3.4: If $A = (a_{ij})$ is an $n \times n$ complex matrix, then

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}, \quad (1.9)$$

is the *spectral norm* of the matrix A, [26].

Definition 1.3.5: Let $A = (a_{ij})$ an $n \times n$ complex matrix with eigenvalues λ_i, $1 \le i \le n$. Then

$$\rho(A) = \max_i |\lambda_i|, \quad (1.10)$$

is the *spectral radius* of the matrix A, [26].

Another notation which is used throughout, the spectrum $\sigma(A)$ of a square matrix A is the set of all eigenvalues of A.

Geometrically, if all the eigenvalues λ_i of A are plotted in the complex z-plane, then $\rho(A)$ is the radius of the smallest disk $|z| \le R$, with center at the origin, which includes all the eigenvalues of the matrix A.

Theorem 1.3.1: For an arbitrary square complex matrix A,

$$\|A\| \ge \rho(A), \quad (1.11)$$

the matrix spectral norms can also be expressed in terms of the spectral radii, [26].