Different trends in graph labelling

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Abstract

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We study four labellings of graphs, divisor labelling, strongly multiplicative labelling, strongly *-labelling and permutation labelling. We give necessary conditions for them and study their independence and whether they are sufficient or not.

Keywords

Graph labelling, divisor labelling, strongly multiplicative labelling, strongly *-labelling and permutation labelling.

Introduction

Introduction

Most graph labeling methods trace their origin to one introduced by Rosa [37]in 1967, or one given by Graham and Sloane [23] in 1980. Rosa [37] called a function f a $\beta - valuation$ of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, \cdots, q\}$ such that, when each edge xy is assigned the label $\mid f(x) - f(y) \mid$, the resulting edge labels are distinct. Golomb [22] subsequently called such labellings graceful and this is now the popular term. Rosa introduced $\beta - valuations$ as well as a number of other labellings as tools for decomposing the complete graph into isomorphic subgraphs. In particular, $\beta - valuations$ originated as a means of attacking the conjecture of Ringel [36] that K_{2n+1} can be decomposed into $2n + 1$ subgraphs that are all isomorphic to a given tree with n edges. Although an unpublished result of Erdős says that most graphs are not graceful (cf. [23]), most graphs that have some sort of regularity of structure are graceful. Sheppard [41] has shown that there are exactly $q!$ gracefully labeled graphs with q edges. Rosa [37] has identified essentially three reasons why a graph fails to be graceful: (1) G has "too many vertices" and "not enough edges," (2) G "has too many edges," and (3) G "has the wrong parity." An infinite class of graphs that are not graceful for the second reason is given in [12]. As an example of the third condition Rosa [37] has shown that if every vertex has even degree and the number of edges is congruent to 1 or 2 (mod 4) then the graph is not graceful. In particular, the cycles C_{4n+1} and C_{4n+2} are not graceful. Acharya [1] proved that every graph can be embedded as an induced subgraph of a graceful graph and a connected graph can be embedded as an induced subgraph of a graceful connected graph. Acharya, Rao, and Arumugam [4] proved: every triangle-free graph can be embedded as an induced subgraph of a triangle-free graceful graph; every planar graph can be embedded as an induced subgraph of a planar graceful graph; and every tree can be embedded as an induced subgraph of a graceful tree. These results demonstrate that there is no forbidden subgraph characterization of these particular kinds of graceful graphs.

Harmonious graphs naturally arose in the study by Graham and Sloane

[23] of modular versions of additive bases problems stemming from error-correcting codes. They defined a graph G with q edges to be *harmonious* if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x) + f(y)$ $(modq)$, the resulting edge labels are distinct. When G is a tree, exactly one label may be used on two vertices. Analogous to the "parity" necessity condition for graceful graphs, Graham and Sloane proved that if a harmonious graph has an even number of edges q and the degree of every vertex is divisible by 2^k then q is divisible by 2^{k+1}. Thus, for example, a book with seven pages (i.e., the cartesian product of the complete bipartite graph $K_{1,7}$ and a path of length 1) is not harmonious. Liu and Zhang [33] have generalized this condition as follows: if a harmonious graph with q edges has degree sequence $d_1, d_2, \cdots, d_p$ then $\gcd(d_1, d_2, \cdots, d_p, q)$ divides $\frac{q(q-1)}{2}$.

They have also proved that every graph is a subgraph of a harmonious graph. More generally, Sethuraman and Elumalai [40] have shown that any given set of graphs $G_1, G_2, \cdots, G_t$ can be embedded in a graceful or harmonious graph. Determining whether a graph has a harmonious labeling was shown to be NP-complete by Auparajita, Dulawat, and Rathore in 2001 (see [32]).

Over the past three decades in excess of 1000 papers have spawned a bewildering array of graph labeling methods. Despite the unabated procession of papers, there are few general results on graph labellings. Indeed, the papers focus on particular classes of graphs and methods, and feature ad hoc arguments. In part because many of the papers have appeared in journals not widely available, frequently the same classes of graphs have been done by several authors and in some cases the same terminology is used for different concepts. In this article, we survey what is known about numerous graph labeling methods. The author requests that he be sent preprints and reprints as well as corrections for inclusion in the updated versions of the survey.

Earlier surveys, restricted to one or two labeling methods, include [10], [13], [31], [19], and [20]. The book edited by Acharya, Arumugam, and Rosa [2] includes a variety of labeling methods that we do not discuss in this survey. The extension of graceful labellings to directed graphs arose in the characterization of finite neofields by Hsu and Keedwell [28], [29]. The relationship between graceful digraphs and a variety of algebraic structures including cyclic difference sets, sequenceable groups, generalized complete mappings, near-complete mappings, and neofields is discussed in [16] and [17]. The connection between graceful labellings and perfect systems of difference sets is given in [11].

"Graph labelling at its heart, is a strong communication between number theory and structure of graphs". Graph labellings were first introduced in the late 1960s. Over the past three decades in excess of 800 papers have spawned a bewildering array of graph labelling methods. Despite the unabated procession of papers, there are a few general results on graph labellings. Labelled graphs serve as useful models for a broad range of applications.

Graph labellings, where the vertices and edges are assigned real values or subsets of a set are subject to certain conditions, have often been motivated by their utility to various applied fields and their intrinsic mathematical interest (logical-mathematical). Graph labellings were first introduced in mid sixties. An enormous body of literature has grown around the subject, especially in the last thirty years or so and is still getting embellished due to increasing number of application driven concepts (See [27]).

Labelled graphs are becoming an increasingly useful family of mathematical models for a broad range of applications. The qualitative labellings of graph elements have inspired research in diverse fields of human enquiry such as conflict resolution in social psychology, electrical theory and energy crisis. Quantitative labellings of graphs have led to quite intricate fields of applications such as Coding Theory problems, including the design of good radar location codes, synch-set codes, missile guidance codes and convolution codes with optimal auto-correlation properties. Labelled graphs have also been applied in determining ambiguities in X-Ray Crystallographic analysis, to design communication network addressing systems, to determine optimal circuit layouts and radio-astronomy, astronomy, circuit design, communication network addressing, data base management, and models for constraint programming over finite domains see [14], [15], [45], [35], [43], [44], and [34] for details. Terms and notation not defined below follow that used in [18] and [19]. Besides its application in mathematics itself.

Throughout this thesis, we use the basic notations and conventions in graph theory as in [25], and in number theory as in [26], [24] and [42].

Summary

Summary

This thesis consists of five chapters.

Chapter one:

In chapter one we introduce some basic definitions and theorems in number theory and graph theory which we need afterwards.

Chapter two:

In chapter two, we study some necessary conditions for a graph to be a divisor graph. Also, we study the dependance of these conditions pair-wisely. And finally we prove that they are altogether not sufficient for a graph to be a non-divisor graph.

Chapter three:

In chapter three, we study some necessary conditions for a graph to be strongly multiplicative. Also, we study the dependence of these conditions pair-wisely. And finally we prove that they are altogether not sufficient for a graph to be a non-strongly multiplicative graph.

Chapter four:

In chapter four, we study some necessary conditions for a graph to be a strongly *-graph. Also, we study the dependence of these conditions pair-wisely. And finally we prove that they are altogether not sufficient for a graph to be a non-strongly *-graph.

Chapter five:

In chapter five, we study some necessary conditions for a graph to be a permutation graph. Also, we study the dependence of these conditions pair-wisely. And finally we prove that they are altogether not sufficient for a graph to be a non-permutation graph.

This thesis contains five papers:

1- The results of Chapter 2 appeared in the journal The Egyptian Mathematical Society Vol. 18(2) 2010.

2- The results of Chapter 3 will appear in the international Canadian journal Ars Combinatoria.

3- Some results of chapter 4 will appear in AKCE International Journal of Graphs and Combinatorics.

4- The results of Chapter 5 will appear in the journal The Egyptian Mathematical Society.

5- Some results of chapter 4 are submitted for publication and still under refereeing

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