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Some Uses of Number Theory on Evaluation of Graphs

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CONTENTS

Summary

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This thesis consists of three chapters.

Chapter zero:
In chapter zero we introduce some basic definitions and theorems in graph theory and Number theory which we need afterwards.
Chapter one:
In chapter one, we study: combination, permutation, strong k-combination and strong k- permutation graphs. We introduce some necessary conditions for a graph to be: a combination graph, a permutation graph, a strong k-combination graph and a strong k-permutation graph. We determine all maximal strong k-combination graphs of order ≤ 6 .
Chapter two:
In chapter two, We study: combination and permutation graphs. We introduce some families to be: combination graphs and permutation graphs.

The results of Chapter 1will appear in the international journal Utilita Mathematica and The results of Chapter 2 will appear in the international journal Ars Combinatoria.

Abstract

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A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labelings were first introduced in the late1960s.Graph labeling" at its heart, is a strong communication between "number theory" and "structure of graphs"

In the intervening years dozens of graph labelings techniques have been studied in over 1000 papers.

Keywords

Graph labeling, combination, permutation, strong k-combination and strong k- permutation graphs.

Introduction

Introduction

Most graph labeling methods trace their origin to one introduced by Rosa [15] in 1967, or one given by Graham and Sloane [7] in 1980. Rosa [15]called a function f a β-valuation of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. Golomb [6] subsequently called such labelings graceful and this is now the popular term. Rosa introduced β-valuations as well as a number of other labelings as tools for decomposing the complete graph into isomorphic subgraphs. In particular, β-valuations originated as a means of attacking the conjecture of Ringel [14] that K_{2n+1} can be decomposed into $2n + 1$ subgraphs that are all isomorphic to a given tree with n edges. Sheppard [21] has shown that there are exactly $q!$ gracefully labeled graphs with q edges. Rosa [15]has identified essentially three reasons why a graph fails to be graceful: (1) G has "too many vertices" and "not enough edges," (2) G "has too many edges," and (3) G "has the wrong parity." An infinite class of graphs that are not graceful for the second reason is given in [1]. As an example of the third condition Rosa [15]has shown that if every vertex has even degree and the number of edges is congruent to 1 or 2 (mod 4) then the graph is not graceful. In particular, the cycles C_{4n+1} and C_{4n+2} are not graceful.

Harmonious graphs naturally arose in the study by Graham and Sloane [7] of modular versions of additive bases problems stemming from error-correcting codes. They defined a graph G with q edges to be harmonious if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x) + f(y)$ (mod q), the resulting edge labels are distinct. When G is a tree, exactly one label may be used on two vertices. Analogous to the “parity” necessity condition for graceful graphs, Graham and Sloane proved that if a harmonious graph has an even number q of edges and the degree of every vertex is divisible by 2^k then q is divisible by 2^{k+1}. Liu and Zhang [11]have generalized this condition as follows: if a harmonious graph with q edges has degree sequence $d_1, d_2, \dots, d_p$ then gcd$(d_1, d_2, \dots d_p, q)$ divides $q(q-1)/2$. They have also proved that every graph is a subgraph of a harmonious graph. Determining whether a graph has a harmonious labeling was shown to be NP-complete by Auparajita, Dulawat, and Rathore in 2001 (see [10]).

Permutation, combination ,strong k-combination and strong k-permutation graphs naturally arose in the study by Hegde and Shetty[9,5].They define a graph G with n vertices to be a permutation graph if there exists an injection f from the vertices of G to $\{1,2,3,\dots,n\}$ such that the induced edge function g_f defined as $g_f(uv) = \frac{f(u)!}{|f(u)-f(v)|!}$ is injective. They say a graph G with n vertices to be a combination graph if there exists an injection f from the vertices of G to $\{1,2,3,\ldots,n\}$ such that the induced edge function g_f defined as $g_f(uv) = \frac{f(u)!}{|f(u)-f(v)|!f(v)!}$ is injective. They prove: K_n is a permutation graph if and only if $n \leq 5$; K_n is a combination graph if and only if $n \leq 2$; C_n is a combination graph for $n > 3$, $k_{n,n}$ is a combination graph if and only if $n \leq 2$; W_n is not a combination graph for $n \leq 6$, and a necessary condition for a (p,q)-graph to be a combination graph is that $4q \leq p^2$ if p is even and $4q \leq p^2 - 1$ if p is odd. They strongly believe that W_n is a combination graph for $n > 6$ and all trees are combination graphs. Baskar Babujee and Vishnupriya [2] prove the following graphs are permutation graphs: P_n ; C_n; stars; graphs obtained adding a pendent edge to each edge of a star; graphs obtained by joining the centers of two identical stars with an edge or a path of length 2; and complete binary trees with at least three vertices.

Seoud and Al-Harere [16] presented two Theorems:(1) A graph $G(n,q)$ having at least 6 vertices , such that 3 vertices are of degree $1, n-1, n-2$ is not a combination graph . (2) A graph $G(n,q)$ having at least 6 vertices, such that there exist 2 vertices of degree $n-3$, two vertices of degree 1and one vertex of degree $n-1$ is not a combination graph. Second ,they show that the following families are combination graphs : Two copies of C_n sharing a common edge, the graph consisting of two cycles of the same order joined by a path of l vertices, the union of three cycles of the same order, the wheel W_n $n \geq 7$, what Hegde and shetty believed, the corona $T_n \odot K_1$, where T_n is the triangular snake , the graph obtained from the gear G_m , by attaching n pendent vertices to each vertex which is not joined to the center of the gear , and some corollaries .

Seoud and Al-Harere [17], prove: the graph $G(n, q), n \geq 3$ is a non-combination graph if it has more than one vertex of degree $n - 1$; and the following graphs are non-combination graphs; G_1+G_2 if n_1 or $n_2 > 2$, n_1 , n_2 $\neq 1$; the double Fan $\overline{K_2} + P_n$; $K_{l,m,n}$; $K_{k,l,m,n}$; $P_2[G]$; $P_3[G]$; $C_3[G]$; $C_4[G]$; $K_m[G]$; $W_m[G]$; the splitting graph of K_n , $S(K_n)$, $n \geq 3$; $K_n - e$, $n \geq 4$; $K_n - 3e$, $n \geq 5$; $K_{n,n} - e$, $n \geq 3$.

Seoud and Al-Harere [18], introduce a theorem on bipartite graphs, and some theorems on chains of two and three complete graphs considering when they are combination or non- combination graphs. They show that some families of graphs are combination graphs. Also, they give a survey for trees of order ≤ 10, which are all combination graphs.

Seoud and Mahran [19] give an upper bound of edges of a permutation graph. They introduce some necessary conditions for a graph to be a permutation graph, and they discuss the independence of these necessary conditions. They show that they are altogether not sufficient for a graph to be a permutation graph.

Seoud, and Salim [20], investigate the permutation labelings for graphs of order 69, they conjecture that every bipartite graph is a permutation graph, armed with proving this conjecture for bipartite graphs of order 650.

Labeled graphs serve as useful models for a broad range of applications such as: coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network addressing, data base management, secret sharing schemes, and models for constraint programming over finite domains–see [3], [4], [23], [13], [24], [25], [22] and [12] for details.

Chapter 0