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Introduction

The extension for linear functionals has been studied in locally convex spaces by several authors. In fact, the importance of the further study of the Hahn-Banach theorems arises from its wide applications in several branches of sciences such as mathematics (complex and functional analysis) and physics (Thermodynamics).

Our goal in this thesis is the establishment of the extension for linear functionals. We are concerned with a very important topic in functional analysis which is the Hahn Banach theorem. The Hahn-Banach theorem is an extension theorem for linear functionals with preservation of some certain property. It has a very powerful statement about convex sets, and it forms the foundations of mathematical treatment of optimization.

It also turns out to be a very handly tool for solving many statements of linear algebra, conic duality theory, piecewise approximation of convex functionals, extension of positive linear functionals, and other results from modern control. Throughout this thesis, we will be concerned with real linear spaces.

Our aim in this thesis is to study different methods of extended functionals defined on a subspace to the whole space such that the growth remains under control.
Also, we introduce a new version of the Hahn-Banach theorem and we study the problem of extending linear functionals defined on subspaces of real linear spaces by many different methods.
Moreover, we generalize our results to the case of a finite system of sublinear and superlinear functionals on E.
More precisely, we devote the last part of our results to give a new general version of Hahn-Banach theorem in real linear spaces.

The Hahn-Banach theorem was discovered by HansHahn (1879-1934) in 1927 and returned to be discovered by Stefan Banach (1892-1945)in 1929. This is one of the fundamental results in Functional Analysis whereby a linear functional which is defined on a subspace of a vector space and dominated by a sublinear functional on the entire space has a linear extension to the entire space which is still dominated by a sublinear functional. Roughly speaking the Hahn-Banach theorem asserts that, if we have a linear functional on a subspace of a linear space whose growth can somehow be controlled, then this functional can be extended to the whole space such that the growth remains under control.

In this thesis, we present a new version of the extension form of Hahn-Banach theorem and the classical Hahn-Banach theorem for sublinear functionals. On other words, we will answer the following question: Does there exist a linear functional L defined on E, which extends a linear functional f_0 defined on M, whatever f_0 has no constrains or f_0 has some constrains such as $P(x) \leq f_0(x)$, $f_0(x) \leq P(x)$, $S(x) \leq f_0(x)$, $S(x) \leq f_0(x) \leq P(x)$ and $P(x) \leq f_0(x) \leq S(x)$ for every $x \in M$, where S and P are sublinear and superlinear functional on E respectively?. Also, we show that there exists a linear functional $L : E \to \Re$ on E for every superlinear functional $P : E \to \Re$ such that $P(x) \leq L(x)$ for every $x \in E$, and there exists a linear functional $L : E \to \Re$ on E for every sublinear functional $S : E \rightarrow \Re$ and superlinear functional $P : E \rightarrow \Re$ such that $P(x) \leq S(x)$ for every $x \in E$. Moreover, we show that if we take $P(x) = -S(-x)$ for every $x \in E$, then we obtain immediately the extension form of Hahn-Banach theorem and the classical Hahn-Banach theorem for sublinear functionals as a special case of our work. Finally, we generalize the above results to the case of a finite system of sublinear and superlinear functionals on E. On other words, suppose that S_j are sublinear functionals on E and P_j are superlinear functionals on E for every $j = 1, 2, 3, ..., n$ such that $P_j(x) \leq S_j(x)$ for every $x \in E$ and for every j. Furthermore, let f_0 be a linear functional on M, where M is a subspace of E such that $P_j(x) \leq f_0(x) \leq S_j(x)$ for every $x \in M$ and for every j. Hence in the last section of this thesis, we answer the following question: Under what conditions f_0 has an extended linear functional L on E such that $L|_M = f_0$ and $P_j(x) \leq L(x) \leq S_j(x)$ for every $x \in E$ and for every j. Also, we show that if we take $P_j(x) = -S_j(-x)$ for every $x \in E$ and

$j = 1, 2, 3, ..., n$, then we obtain immediately the extension form of Hahn-Banach theorem for the finite system of sublinear functionals(which states that: If S_j are sublinear functionals on E and f_0 is a linear functional on M, where M is a subspace of E such that $f_0(x) \leq S_j(x)$ for every $x \in M$ and for every j, then there exists a linear functional L on E such that $L|_M = f_0$ and $L(x) \leq S_j(x)$ for every $x \in E$ and for every j).

The thesis consists of three chapters.
Chapter one introduces the basic definitions and facts from functional analysis.

In Chapter two, we answer the following question: Does there exist a linear extension functional L defined on a real vector space E, which extends a linear functional f_0 defined on a linear subspace M of E such that f_0 has no any constrained conditions or f_0 has a certain constrained condition, like as $P(x) \leq f_0(x)$, $f_0(x) \leq P(x)$, $S(x) \leq f_0(x)$, $S(x) \leq f_0(x) \leq P(x)$ and $P(x) \leq f_0(x) \leq S(x)$ for every $x \in M$, where S and P are sublinear and superlinear functional on E respectively?, which considered as new version of the extension form of Hahn-Banach theorem. Moreover, we introduce the classical Hahn-Banach theorem for superlinear functionals, and we generalize the classical Hahn-Banach theorem for sublinear functionals to the classical Hahn-Banach theorem for sublinear and superlinear functionals. Also, we show that if we take $P(x) = -S(-x)$ for every $x \in E$ in our theorem, then we immediately obtain the classical Hahn-Banach theorem for sublinear functionals.

The new results contained in this chapter have been published in proceedings of international conference on a theme " The actual problems of mathematics and computer science" [14].

In Chapter three, we introduce two completely different methods for the proof of last part which was mentioned in the above question in Chapter 2 and we generalize it to the case of a finite system of sublinear and superlinear functionals on E. Suppose that S_j and P_j are a finite systems of sublinear and superlinear functionals respectively, where $j = 1, 2, 3, ..., n$ such that $P_j(x) \leq S_j(x)$ for every $x \in E$ and for every j. Furthermore, let f_0 be a linear functional on M, where M is a subspace of E such that $P_j(x) \leq f_0(x) \leq S_j(x)$ for every $x \in M$ and for every j. Now, we discuss the following question: Under what conditions the functional f_0 has an extension linear functional L on E such that $P_j(x) \leq L(x) \leq S_j(x)$ for every $x \in E$ and for every j. Also, if we take $P_j(x) = -S_j(-x)$ for every $x \in E$ and for every j, then we get a new version of the extension form of Hahn-Banach theorem for a finite system of sublinear functionals [15].

Chapter 1

Basic Concepts

In this chapter we introduce some basic definitions and facts from functional analysis. These preliminaries will be used in the subsequent chapters but from time to time we supplement them with other results which make the discussion more complete.

1.1. List of symbols and notations.

R	The set of all real line
R^n	Eculidean space
E	Real vector space
M	Subspace of real vector space
I	the identity mapping
$\underline{0}$	the zero vector on E
N	the set of all natural number
L	Linear functional on E
S	Sublinear functional on E
P	Superlinear functional on E

K Semi convex cone

f_0 Linear functional on M

$Subl(E)$ The set of all sublinear functionals on E

$Supl(E)$ The set of all sublinear functionals on E

$Lin(E)$ The set of all linear functionals on E

$Lin(M)$ The set of all linear functionals on M

$Homg(E)$ The set of all homogeneous functionals on E

$Fun(E)$ The set of all functionals on E

$T_1 = \{x \in E : P(x) \leq S(x)\}$

$T_2 = \{x \in E : S(x) \leq P(x)\}$

$T_3 = \{x \in E : P(x) = S(x)\}$

$\Gamma_1 = \{L \in Lin(E) : L|_M = f_0$ and $L(x) \leq S(x)$ for every $x \in E\}$

$\Gamma_2 = \{L \in Lin(E) : L|_M = f_0$ and $P(x) \leq L(x)$ for every $x \in E\}$.

S_j A finite system of sublinear functionals on E.

P_j A finite system of superlinear functionals on E.

1.2. Basic definitions and some results.

In this section we present basic definitions and some facts from functional analysis. In the following definitions we suppose that E is a real vector space.

Definition 1.2.1.

The functional $S : E \to \Re$ is called sublinear if it possesses the following properties:

(1)$S(x+y) \leq S(x)+S(y)$ for every $x, y \in E$ (i.e., S is subadditive),

(2)$S(\alpha x) = \alpha S(x)$ for every $x \in E$ and $\alpha > 0$(i.e., S is positively homogeneous).

The set of all sublinear functionals on E is denoted by $Subl(E)$.

Definition 1.2.2.

The functional $P : E \to \Re$ is called superlinear if it possesses the following properties:

(1)$P(x + y) \geq P(x) + P(y)$ for every $x, y \in E$ (i.e., P is superadditive),

(2)$P(\alpha x) = \alpha P(x)$ for every $x \in E$ and $\alpha > 0$ (i.e., P is positively homogeneous).

The set of all superlinear functionals on E is denoted by $Supl(E)$.

Any linear functional on E is clearly sublinear and superlinear. On other words, $Lin(E) = Subl(E) \bigcap Supl(E)$, where $Lin(E)$ is the set of all linear functionals on E. Also, it is easy to verify that if $E \equiv \Re^n, S(x) = \sqrt{\sum_{i=1}^{n} x_i^2} = \|x\|$ and $P(x) = -\sqrt{\sum_{i=1}^{n} x_i^2} + x_n$ for every $x \in \Re^n$, then $S \in Subl(E)$ and $P \in Supl(E)$.

Definition 1.2.3.

The non-empty subset A of E is called convex if $\alpha x + (1-\alpha)y \in A$ for every $x, y \in E$ and $0 \leq \alpha \leq 1$.

Remark 1.2.1.

A subset M of a real vector space E is convex if and only if for every $x_i \in M$, real $\alpha_i \geq 0$, $i = 1, 2, 3, ..., n$, $\sum_{i=1}^{n} \lambda_i = 1$, $\sum_{i=1}^{n} \lambda_i x_i \in M$.

Definition 1.2.4.

The functional $f : A \rightarrow \Re$ is called convex, where A is a non-empty convex subset of E if $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$ for every $x, y \in A$ and $0 \leq \alpha \leq 1$. Also, f is called concave if and only if $-f$ is convex.

It is easy to verify that every sublinear functional is convex but the converse is not always true (For the functional $f(x) = x^2$, where $x \in \Re$ is convex but not sublinear).

Remark 1.2.2.

If f is a convex functional on A, where A is a non-empty convex subset of E, $x_1, x_2, ..., x_n \in A$ and $\alpha_1, \alpha_2, ..., \alpha_n$ are positive real numbers such that $\sum_{i=1}^{n} \alpha_i = 1$, then $f(\sum_{i=1}^{n} \alpha_i x_i) \leq \sum_{i=1}^{n} \alpha_i f(x_i)$.

The above inequality is called Jensen's inequality. Sometimes this relation is taken as the definition of a convex functional [19].

Definition 1.2.5.

Let $S \in Subl(E)$ and A a non-empty convex subset of E, then a function $f : A \rightarrow E$ is said to be S - convex if $S(x + f(\sum_{i=1}^{2} \alpha_i a_i)) \leq S(x + \sum_{i=1}^{2} f(\alpha_i a_i))$ for every $x \in E$, $a_i \in A$, $\alpha_i > 0$ (for $i = 1, 2$) and $\sum_{i=1}^{2} \alpha_i = 1$ [21].

It is easy to see that linear functions, identity functions, inclusion functions and zero functions are clearly S - convex.