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Fixed point theorems in some types of metric-like spaces

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"BESM ELLAH ERRAHMAN ERRAHEEM."

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"No One Beyond Mistake"

Contents

Summary

It is well known that a normed space X is uniformly convex (smooth) if and only if its dual X^* is uniformly smooth (convex). We extend some of these geometric properties to the so-called 2-normed spaces and also we introduce definition of uniformly smooth 2-normed spaces. We get some fundamental links between Lindenstrauss duality formulas. Besides, a duality property between uniform convexity and uniform smoothness of 2-normed space is also given. Moreover, we introduced a definition of Metric projection in a 2-normed space and also theorem of the approximation of fixed points in 2-Hilbert spaces.

1. In **chapter 1**: we give a summary of quotient spaces and state the Hahn-Banach Theorem; which is the one of the fundamental theorems of functional analysis. We study some of the geometric properties in linear normed spaces and we give a brief history of Metric projection existence and uniqueness in different spaces.

2. In **chapter 2**: we discuss some spaces like 2-metric spaces and linear 2-normed spaces. We define a Cauchy sequence and a convergent sequence in both spaces and mention the relation between both concepts and also the notions of bounded bilinear function.

3. In **chapter 3**: we study the completion of linear 2-normed spaces.

4. In **chapter 4**: we study some geometric properties for linear 2-normed spaces like strictly convex, strictly 2-convex, uniformly convex and uniformly 2-convex 2-Banach spaces. We give new definitions and prove new results. We prove an important result which state that: "a 2-normed space X is uniformly convex if and only if all dual spaces $(X/V(c))^*$ or X^*_c for all $c \neq 0$ in X are uniformly smooth". It is our purpose to extend the notion of

uniform smoothness for Banach spaces to 2-normed spaces. Moreover, by extending some theorems for uniformly convex uniformly smooth Banach spaces to the case of 2-normed spaces, we prove the existence of the metric projection point in uniformly convex 2-normed spaces and also the fixed point theorem of non-self maps in 2-Hilbert spaces.

Introduction

In 1963, S. Gähler [17, 18] published the first of several papers entitled “2-metric spaces and their topological structures” dealing with spaces on which is defined what he calls 2-metric space. The second article by S. Gähler [21] entitled “Linear 2-normed spaces” is limited to studying the special class of 2-metric spaces which are linear and have defined on them a 2-norm space. A. White [22] extended the concept to 2-Banach spaces, where White established Hahn-Banach theorem in a 2-Banach space.

The concept of subbasis that forms a topological space has also been considered by Gähler [17, 18]. He showed that by suitably defining the members of the subbasis, a topology can be considered in a 2-metric space.

Since 1963, C.R. Dimminie, R.W. Freese, S. Gähler, C.S. lin, A. White, M.E. Newton and many others have developed extensively the geometric structure of linear 2-normed spaces. C.R. Dimminie, S. Gähler and A. White introduced the concept of strictly convex, 1974, and also strictly 2-convex, 1979, in 2-normed linear spaces to the sitting of linear 2-normed spaces and gave some characterizations of strictly convex and strictly 2-convex. In 1969, M.E. Newton constructed a parallel definition of uniformly convex in linear 2-normed spaces and C.S. lin, 1992, introduced the concept of uniformly 2-convex in linear 2-normed spaces and gave some relation between uniformly 2-convex and strictly 2-convex in linear 2-normed spaces. (see [26, 27, 32, 34])

Chapter 1

Basic Notions

1.1 Hahn-Banach theorems

The Hahn-Banach theorem is the one of the most important theorems in functional analysis. To state it, we need the following definitions:

Definition 1.1 *[1]*
Let X be a linear space (not necessarily normed), and consider the mapping $p : X \to \mathbb{R}$ with the following properties:

(i) $p(x) \geq 0$ for all $x \in X$,

(ii) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,

(iii) $p(\alpha x) = \alpha p(x)$ for all $x \in X$ and scalar α, $\alpha > 0$.

A mapping satisfying the all above three conditions is called a *convex functional*. A mapping satisfying condition (*ii*) is called *subadditive* and is called *positive homogeneous* if it satisfies (*iii*). If p satisfies (*ii*) and (*iii*), then p is called a *sublinear functional*.

Definition 1.2 *[1]*
Let C be a proper subset of a real vector space X. Let f be a map defined on the subset C into a vector space W. The mapping $f : C \to W$ (defined on C into W) is said to be extended to X if there exists a

map $F : X \to W$ (defined on X into W) such that $F(x) = f(x)$ for all $x \in C$. The map F is called an *extension of f from C to X.*

Theorem 1.3 *[1, 2](Hahn Banach Theorem, Analytic Form)*
Let M be a subspace of a real linear space V, $p : V \to \mathbb{R}$ be a sublinear functional on V and $f : M \to \mathbb{R}$ be a linear functional defined on M such that

$$f(x) \leq p(x) \quad \forall x \in M.$$

Then,

(i) there exists a linear functional F defined on V which is an extension of f,

(ii) $F(x) \leq p(x)$ for all $x \in V$.

Theorem 1.4 *[1, 2]*
Let f be a bounded linear functional defined on a subspace M of a real normed linear space X. Then,

(i) there exists a bounded linear functional F defined on X, which extends f,

(ii) $\|F\| = \|f\|$.

Theorem 1.5 *[1]*
Let X be a normed linear space and let $x_o \neq 0$ be an arbitrary element of X. Then, there exists a bounded linear functional f on X such that

$$\|f\| = 1 \text{ and } f(x_o) = \|x_o\|.$$

Theorem1.6 *[1]*
Let C be a subspace of a normed linear space X. Let $x_0 \notin C$ such that

$$\delta := \inf_{x \in C} \|x - x_o\| > 0.$$

Then, there exists $f \in X^*$ such that

(i) $f(x_o) = \delta$,

(ii) $\|f\| = 1$,

(iii) $f(x) = 0$ for all $n \in C$.

1.2 Quotient spaces

Definition 1.7 *[3, 4]*
Let V be a vector space and W be a subspace of V. For each $v \in V$, the set $[v] = v + W := \{v + w : w \in W\}$ is called a *coset of W in V* or *an affine subspace of V*. The set of all cosets of W is called the *quotient space of V by W*, denoted by V/W (read V mod W).

Theorem 1.8 *[3, 4]*
Let V be a vector space over a field F and W be a subspace of V. The quotient space V/W becomes a vector space over F if we let $[0] := 0+W$ be the zero vector and define addition and scalar multiplication by

1. $[u] + [v] := [u + v]$,
2. $[\alpha u] := \alpha[u]$ *for all* $u, v \in V$ *and* $\alpha \in F$.

Theorem 1.9 *[3, 4]*
Let V be a vector space over a field F and W be a subspace of V. The map $\pi : V \to V/W$ defined by

$$\pi(v) := v + W = [v] \quad \forall v \in V,$$

is a surjective linear transformation with $\ker\pi = W$.

The map $\pi : V \to V/W$ defined in above Theorem is called the canonical map (or canonical projection or natural projection) from V onto V/W.

Theorem 1.10 *[3, 4]*
W be a subspace of a vector space V. If any two of the vector spaces V, W and V/W are finite-dimensional, then all three spaces are finite-dimensional and

$$\dim V/W = \dim V - \dim W.$$

Theorem 1.11 *[4]*
Let $(X, \|.\|)$ be a linear normed space and Y be a subspace of X, then X/Y is a normed vector space with

$$\|[x]\|_{X/Y} := \inf_{y \in Y} \|x + y\|.$$

Theorem 1.12 *[4]*
Let X be a Banach space and Y be a closed subspace of X, then X/Y is a Banach space.

1.3 Geometric properties in normed spaces

Definition 1.13 *[5, 6, 7]*
A normed space X is said to be *strictly convex* if for all $x, y \in X$ with $x \neq y$ and $\|x\| = \|y\| = 1$, then $\|\lambda x + (1-\lambda)y\| < 1$ for all $\lambda \in (0, 1)$.

This mains that the midpoint $(x+y)/2$ of two distinct points x and y in the unit sphere $S_X := \{x \in X : \|x\| = 1\}$ of X does not lie on S_X. In other words, if $x, y \in S_X$ with $\|x\| = \|y\| = \|(x+y)/2\|$, then $x = y$.

Example 1.14 Consider $X = \mathbb{R}^n$, $n \geq 2$ with norm $\|x\|_2$ defined by $\|x\|_2 := \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Then, X is strictly convex. However, the same space X with different norm $\|x\|_1$, defined by $\|x\|_1 := |x_1| + |x_2| + \ldots + |x_n|$, is not strictly convex.

Proposition1.15 *[5]*
Let X be a Banach space. Then, the following are equivalent:

(a) X is strictly convex.

(b) For each nonzero $f \in X^*$, there exists at most one point x in X with $\|x\| = 1$ such that $\langle x, f\rangle = f(x) = \|f\|$.

Proposition 1.16 *[5]*
Let X be a Banach space. Then, the following are equivalent:

(a) X is strictly convex.

(b) For every $1 < p < \infty$, $\|tx + (1-t)y\|^p < t\|x\|^p + (1-t)\|y\|^p$ for all $x, y \in X$, $x \neq y$ and $t \in (0, 1)$.

Theorem 1.17 *[8]*
Let $(E, \|.\|_1)$ be a linear normed space. Let A be a continuous and one-to-one linear mapping from $(E, \|.\|_1)$ onto a strictly convex normed space $(F, \|.\|_F)$.Then, $(E, \|.\|_2)$ is a strictly convex normed space, where $\|x\|_2 = \|x\|_1 + \|Ax\|_F$ for all $x \in E$.

Definition 1.18 *[5, 6, 7]*
A normed space X is called *uniformly convex* if for any $\varepsilon \in (0, 2]$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in X$ with $\|x\| = 1$, $\|y\| = 1$ and $\|x - y\| \geq \varepsilon$, then $\|\frac{1}{2}(x + y)\| \leq 1 - \delta$.

This means that if x and y are two elements in the closed unit ball $B_X := \{x \in X : \|x\| \leq 1\}$ with $\|x - y\| \geq \varepsilon > 0$, then the midpoint of x and y lies inside the unit ball B_X at a distance of at least δ from the unit sphere S_X.

Example 1.19 Every real inner product space H is uniformly convex. The spaces ℓ_p and $L_p[a, b]$, $1 < p < \infty$, are uniformly convex. However, the spaces ℓ_1, ℓ_∞, c, c_0, $L_1[a, b]$, $L_\infty[a, b]$ and $C[a, b]$ are not.

One can see that $\mathbb{R}^2$ with norm $\|(x_1, x_2)\|_2 = (x_1^2 + x_2^2)^{1/2}$ is uniformly convex, while the same space $\mathbb{R}^2$ with each of the two norms $\|(x_1, x_2)\|_1 = |x_1| + |x_2|$ and $\|(x_1, x_2)\|_\infty = \max\{|x_1|, |x_2|\}$ is not uniformly convex. Consequently, uniform convexity of a normed space X is actually a property of the norm on X. In addition, every uniformly convex normed space is strictly convex. but the converse is not true.

Example 1.20 Let $\beta > 0$ and let $X = c_o$ with the norm $\|.\|_\beta$ defined by

$$\|x\|_\beta := \|x\|_{c_o} + \beta \left(\sum_{i=1}^{\infty} \left(\frac{x_i}{i}\right)^2 \right)^{\frac{1}{2}}.$$

The spaces $(c_o, \|.\|_\beta)$, for $\beta > 0$, are strictly convex, but not uniformly convex.

Proposition 1.21 *[5, 7]*
Let X be a uniformly convex Banach space. Then, we have the following:

(a) *For any r and ε, with $r \geq \varepsilon > 0$, and for any elements $x, y \in X$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x - y\| \geq \varepsilon$, there exists a $\delta = \delta(\varepsilon/r) > 0$ such that*

$$\|(x + y)/2\| \leq r[1 - \delta(\varepsilon/r)].$$

(b) *For any r and ε, with $r \geq \varepsilon > 0$, and for any elements $x, y \in X$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x - y\| \geq \varepsilon$, there exists a $\delta = \delta(\varepsilon/r) > 0$ such that*

$$\|tx + (1 - t)y\| \leq r[1 - 2\min\{t, 1 - t\}\delta(\varepsilon/r)] \;\forall t \in (0, 1).$$