



Faculty of Education

Department of Mathematics

Exact and Computational Methods for Solving some Integral and Differential Equations

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(Pure Mathematics)

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

(قَالُوا سُبْحَانَكَ لَا عِلْمَ لَنَا إِلَّا مَا عَلَّمْتَنَا إِنَّكَ أَنْتَ الْعَلِيمُ الْحَكِيمُ)

الإهداء

إن الحمد لله نحمده ونستعينه ونستغفره ونستهديه، ونصلي ونسلم على المبعوث رحمة للعالمين سيدنا محمد بن عبد الله عليه أفضل الصلاة والسلام. تعتبر الرسالة الإلهية لرسول الله محمد (صلي الله عليه وسلم) تحريراً للعقول من الخرافات ودعوة للتأمل واكتشاف أسرار الكون " سَنُرِيهِمْ آيَاتِنَا فِي الْآفَاقِ وَفِي أَنْفُسِهِمْ حَتَّىٰ يَتَبَيَّنَ لَهُمْ أَنَّهُ الْحَقُّ أَوَلَمْ يَكْفِ بِرَبِّكَ أَنَّهُ عَلَىٰ كُلِّ شَيْءٍ شَهِيدٌ". [فصلت:53] وهناك العديد من النصوص التي تحت علي العلم وطلبه، لذا فإني أسأل الله أن يعلمنا ما ينفعنا وأن ينفعنا بما علمنا، وأن ينفع بهذا العمل.

يقول العالم (الفيلسوف) جاليليو:

(إن الرياضيات هي اللغة التي إذا فهمناها لأصبح في استطاعتنا فهم أسرار الكون).

وبهذه المناسبة يسعدني أن أهدي هذه الرسالة التي أسأل الله أن يتقبلها وينفع بها إلى:

- مدرستي الأولى التي علمتني العطاء بلا مقابل، وغمرتني بالحنان والحب الخالصين، وتحملت صعاب الحياة عن طيب نفس، بارك الله في عمرك يا أمي وألبسك ثياب الصحة والعافية، وأحسن خاتمتك، ورزقك الفردوس الأعلى.
- من علمني الصمود مهما تبدلت الظروف إلى أبي - رحمه الله - الذي لولا تشجيعه وتحفيزه ما اكتمل هذا العمل جعله الله في ميزان حسناته ورزقه الفردوس الأعلى.
- من شملوني بالعطف، وأمدوني بالعون، وحفزوني للتقدم، وحبهم يجري في عروقي، إخوتي وأخواتي رعاهم الله.
- أساتذتي الكرام، نجوم الهدى في ليل الظلام، من كلت أناملهم ليمهدوا لي الطريق إلى الأمام، فلهم مني جزيل الشكر والتقدير والاحترام.
- زملائي الأوفياء، من تجسدت فيهم معاني الحب والوفاء والصدق والعطاء.
- وأخيراً أدعو الله- سبحانه وتعالى- أن يتقبل هذا الجهد وأن يجعله زخراً لي يوم القيامة وأن ينفع به المسلمين.

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بداية أشكر من تفضل وتكرم، وأعطى وأنعم، ووفق ويسر، خالقي ورازقي وولي نعمتي ربي ورب كل شيء.

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يارة مصطفى

Abstract

This thesis consists of five chapters distributed as follows:

Chapter I

This chapter is an introduction to the basic concepts of integral equations. It includes the classifications of integral and integro-differential equations. It also includes an introduction to the basic definitions and the necessary properties for fractional calculus such as the definitions of Riemann-Liouville, Caputo and discusses some necessary mathematical definitions that will arise in the study of these concepts.

An introduction to the various methods used in this thesis to obtain the exact solutions and the numerical solutions. It also includes an introduction to some theorems and basic concepts of measure theory which will be used in this thesis.

Chapter II

In this chapter, we improve and extend variational iteration method (VIM) and Chebyshev spectral method to find the exact solutions and the approximate solutions for fractional differential equations, fractional integro-differential equations, nonlinear systems of fractional integro-differential equations and generalized Abel's integral equations of the second kind. Moreover, we aim to study the convergence of the VIM for fractional differential equations, fractional integro-differential equations, nonlinear systems of fractional integro-differential equations and generalized Abel's integral equations of the second kind and to address the sufficient condition for convergence. The results obtained by variational iteration method and Chebyshev spectral method in this chapter are compared with the exact solutions and with the results obtained by some other authors, this comparison shows that we obtained better results and more accurate.

Chapter III

In this chapter, we apply the differential transform method (DTM) and homotopy perturbation method (HPM) to solve fifth-order boundary value problem, system of second-order boundary value problem, system of Volterra integral equations, systems of linear and nonlinear integro-differential, Cauchy problem, boundary value problem of fractional order, fractional integro-differential equations and nonlinear systems of fractional integro-differential equations.

In addition, we extend the modified Laplace decomposition method (mLDM) and the modified Laplace decomposition method with the Padé approximant (mLD-PA) to solve boundary value problem of fractional order and systems of linear and nonlinear fractional integro-differential equations. The results obtained by differential transform method, fractional differential transform method, homotopy perturbation method,

modified homotopy perturbation method, modified Laplace decomposition method and modified Laplace decomposition method with the Padé approximant in this chapter are compared with the exact solutions and with the results obtained by some other authors, this comparison shows that we obtained better results and more accurate.

Chapter IV

In this chapter, we extend and generalize the Haar wavelet method (HWM) and Legendre wavelets method (LWM) to solve systems of Fredholm and Volterra integro-differential equations of the second kind, higher-order boundary value problems, fractional integro-differential equations and systems of Fredholm and Volterra fractional integro-differential equations of the second kind. Study problems are performed to test the applicability, efficiency and accuracy of this method. The results obtained by Haar wavelet method and Legendre wavelets method in this chapter are compared with the exact solutions and with the results obtained by some other authors, this comparison shows that we obtained better results and more accurate.

Chapter V

In this chapter, we consider the kernels of Baskakov--Durrmeyer and the Szász--Mirakjan--Durrmeyer operators. We establish a Bernstein type inequality for these operators and apply the results to the quasi-interpolants. For the Baskakov--Durrmeyer quasi-interpolants, we give a representation as linear combinations of the original Baskakov--Durrmeyer operators and prove an estimate of Jackson--Favard type and a direct theorem in terms of an appropriate K-functional. Also, we present the Szász--Mirakjan--Durrmeyer operator with respect to an arbitrary measure in the one-dimensional case.

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