Faculty of Education
Mathematics Department

IDEALS OF SEQUENCES AND OPERATORS

A Thesis

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(Functional Analysis)

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Contents

List of Abbreviations and Symbols

We list up basic notations, abbreviations and symbols to be used in the sequel:

1. $\mathbb{R}$: is the field of real numbers.

2. $\mathbb{Z}$: is the set of integers.

3. $\mathbb{N} = \{1, 2, 3, ...\}$: is the set of natural numbers.

4. Π: is the set of all permutations of the set $\mathbb{N}$.

   For two Banach spaces X and Y:

5. $\mathcal{B}(X, Y) = \{T; T : X \to Y\}$: is the space of all bounded (continuous) linear operators from X into Y endowed with the usual norm

$$||T|| = \sup_{||x|| \leq 1} ||T(x)|| = \sup_{||x||=1} ||T(x)|| = \sup_{x \neq 0} \frac{||T(x)||}{||x||},\ x \in X.$$

6. $\mathcal{B}(X)$ is written for $\mathcal{B}(X, X)$.

7. $\mathfrak{F}(X, Y)$: is the space of all finite dimensional operators from X into Y.

8. $\mathcal{K}(X, Y)$: is the space of all compact operators from X into Y.

9. $\mathbf{N}(X,Y)$: is the space of all nuclear operators from X into Y.

For every operator $T \in \mathcal{B}(X,Y)$:

10. $\mathcal{D}(T) \subset X$: is the domain of T.
11. $\mathcal{R}(T) := \{y \in Y : y = T(x) \ for \ some \ x \in \mathcal{D}(T)\}$: is the range of T.
12. $\mathcal{N}(T) := \{x \in \mathcal{D}(T) : T(x) = 0\}$: is the null space (or the kernel) of T.

For a Banach space X:

13. $X^* = \mathcal{B}(X, \mathbb{R})$: is its dual (or conjugate) space.
14. I_X: is the identity operator of X.
15. $S_X = \{x \in X : ||x|| = 1\}$: is the unit sphere of X.
16. $B_X = \{x \in X : ||x|| \leq 1\}$: is the unit ball of X.

For any subset M of X:

17. span M: is the set of all finite linear combinations
$$\sum_i^n r_i m_i, \ (m_i \in M, \ r_i \in \mathbb{R}, \ i = 1, 2, 3, ..., \ n \in \mathbb{N}).$$
18. $e_n = (0, 0, ..., 1, 0, 0, ...)$: is the unit vector of a linear space, where 1 appears at n^{th} place for all $n \in \mathbb{N}$.
19. $\theta = (0, 0, 0, ...)$: is the zero vector of a linear space.
20. $card(x)$: is the cardinality of $\{n : x_n \neq 0\}$ for any finite sequence $x = (x_n)$.
21. I: is the set of all bounded scalar sequences.
22. ℓ_0: is the space of all real or complex-valued sequences.

23. ℓ_p, $1 \leq p < \infty$: is the space of all absolutely p-summable sequences $x = (x_k)$ with real or complex terms such that $\sum_{k=1}^{\infty} |x_k|^p < \infty$ normed by

$$||x|| = (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}}.$$

24. ℓ_p^n, $1 \leq p < \infty$, $n \in \mathbb{N}$: is the space of all n-tuples absolutely p-summable sequences $x = (x_k)$ with real or complex terms normed by

$$||x|| = (\sum_{k=1}^{n} |x_k|^p)^{\frac{1}{p}}.$$

25. ℓ_∞: is the space of all bounded sequences $x = (x_k)$ with real or complex terms normed by $\|x\|_\infty = \sup_k |x_k|$, where $k \in \mathbb{N}$.

26. c: is the space of all convergent sequences $x = (x_k)$ with real or complex terms normed by $\|x\|_\infty = \sup_k |x_k|$, where $k \in \mathbb{N}$.

27. c_0: is the space of all null sequences $x = (x_k)$ with real or complex terms normed by $\|x\|_\infty = \sup_k |x_k|$, where $k \in \mathbb{N}$.

$$c_0 \subset c \subset \ell_\infty.$$

Summary

The aim of this thesis is to study some operator ideals according to that their sequences of s-numbers belong to certain sequence ideal. For example if the sequence of approximation numbers of an operator converges to zero then it belongs to the ideal of compact operators and if this sequence is absolutely p-summing, then it belongs to the ideal of Shatten Von Neumann operators.

The thesis consists of five chapters:

Chapter 1

This chapter is an introductory chapter. It contains definitions and basic concepts that are used throughout this thesis. It is regarded as a short survey of the basic needed material.

Chapter 2

Our goal in this chapter is to discuss the two concepts of operator ideal and sequence ideal. The purpose of it is to present a short survey of some needed definitions and basic concepts of these two important vital topics: operator ideal and sequence ideal.

Chapter 3

The aim of this chapter is studying the ideal of bounded linear

operators between arbitrary Banach spaces whose approximation numbers sequence belongs to the sequence space defined by a sequence of modulus functions. As a special case of our results, we form an operator ideal using some well-known spaces like Cesáro sequence space and Orlicz sequence space. In addition, we prove that the finite rank operators are dense in the operator ideal formed by those spaces. Finally, we show that the components of the operator ideal defined by them are complete. Our results generalize those in [27] by Faried and Bakery.

Some results of this chapter are:
Under submission [31].

Chapter 4

In this chapter, we introduce new generalized fractional order difference sequence spaces which are defined by a sequence of modulus functions. Different algebraic and topological properties of these spaces like linearity, completeness and solidity, etc are studied. Furthermore, we drive necessary and sufficient conditions for the inclusion relations involving these spaces. Also, we prove that the ideal of bounded linear operators between arbitrary Banach spaces whose approximation numbers sequence belongs to those spaces can't be obtained anyway because they are not solid.

Some results of this chapter are published in:
Mathematical Sciences Letters, V. 6 N. 2, 2017 [29].

Chapter 5

This chapter is devoted to examine some general properties of the generalized fractional order difference sequence spaces defined by a sequence of Orlicz functions. In addition, we give some inclusion theorems of these spaces. Furthermore, we show that the ideal of bounded linear operators between arbitrary Banach spaces whose approximation numbers sequence belongs to those spaces can't be obtained anyway since they are not solid.

Some results of this chapter are accepted in:
International Journal of Advancement in Engineering Technology, Management and Applied Science [30].

Introduction

Functional Analysis is one of the most central and important subjects in the field of pure mathematics. In fact, it depends on every abstract branches of mathematics such as Linear Algebra, Mathematical Logic, Real Analysis, Measure Theory and Topology. So it has several applications in other branches of mathematics which deal with real problems, this happens by using techniques of functionals.

Two of very important vital topics in Functional Analysis field are operator ideal and sequence ideal. The hypothesis of operator ideal goals has an uncommon importance in useful examination as a result of the wide applications in geometry of Banach spaces, fixed point hypothesis, spectral hypothesis and hypothesis eigenvalue dispersions, etc. A large part of the administrator goals in the family of Banach spaces or normed spaces in direct practical examination are determined by diverse scalar grouping spaces.

Kizmaz [39] defined the difference sequence spaces $X(\Delta) = \{x = (x_k) \in \ell_0 : (\Delta x_k) \in X\}$, for $X \in \{\ell_\infty, c, c_0\}$, where $\Delta x_k = (x_k - x_{k+1})$. During the last 35 years, a lot of results have been found by many mathematicians satisfying more various generalizations of difference sequence spaces defined by Kizmaz. First generalization was introduced by Colak and Et [17] who defined the sequence spaces $X(\Delta^m) = \{x = (x_k) \in \ell_0 : (\Delta^m x_k) \in X\}$, for $X \in \{\ell_\infty, c, c_0\}$, where $m \in \mathbb{N}$ and $\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \begin{pmatrix} m \\ i \end{pmatrix} x_{k+i}$. Later, Et and Esi [25] defined the sequence spaces $X(\Delta^m_\nu) = \{x = (x_k) \in \ell_0 : (\Delta^m_\nu x_k) \in X\}$, for $X \in \{\ell_\infty, c, c_0\}$, where $\Delta_\nu^m x_k = \sum_{i=0}^m (-1)^i \begin{pmatrix} m \\ i \end{pmatrix} \nu_{k+i} x_{k+i}$, where $\nu = (\nu_k)$ is any fixed sequence of non-zero complex numbers. After that, Baliarsingh and Dutta [10, 21] unified most of the difference sequence spaces defined earlier and extended these results to the fractional case. For a positive proper fraction α and a bounded sequence of positive reals (p_k), they introduced the fractional order difference sequence spaces $X(\Delta^\alpha, (p_k)) = \{x = (x_k) \in \ell_0 : (\Delta^\alpha x_k) \in X((p_k))\}$ for $X \in \{\ell_\infty, c, c_0\}$, where Δ^α is called the fractional order difference operator and defined by

$$\Delta^\alpha x_k = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k+i}.$$

For a modulus f, Maddox [55] and Ruckle [69] defined the sequence spaces $X(f) = \{x = (x_k) \in \ell_0 : (f(|x_k|)) \in X\}$, for $X \in \{\ell_\infty, c, c_0\}$. Kolk [41, 42] gave an extension of these sequence spaces by considering a sequence of modulus functions (f_k) and defined the sequence spaces $X((f_k)) = \{x = (x_k) \in \ell_0 : (f_k(|x_k|)) \in X\}$, for $X \in \{\ell_\infty, c, c_0\}$. After that, Gaur and Mursaleen [32] defined the sequence spaces
$X((f_k), \Delta) = \{x = (x_k) \in \ell_0 : (f_k(|\Delta x_k|)) \in X\}$, for $X \in \{\ell_\infty, c, c_0\}$, for any sequence of modulus functions (f_k). Recently, Khan [35, 38] defined, for any sequence of real numbers (p_k) and any sequence of modulus functions, the sequence spaces
$X((f_k), (p_k)) = \{x = (x_k) \in \ell_0 : (f_k(|x_k|)) \in X((p_k))\}$
and $X((f_k), (p_k), \Delta_\nu^m) = \{x = (x_k) \in \ell_0 : (\Delta_\nu^m x_k) \in X((f_k), (p_k))\}$,
for $X \in \{\ell_\infty, c, c_0\}$, where $\Delta_\nu^m x_k = \sum_{i=0}^m (-1)^i \begin{pmatrix} m \\ i \end{pmatrix} \nu_{k+i} x_{k+i}$.

Lindenstrauss and Tzafriri [49] used the idea of Orlicz function to construct the sequence space
$\ell_\phi = \{x = (x_k) \in \ell_0 : \sum_{k=1}^\infty \phi(\frac{|x_k|}{\rho}) < \infty \textit{ for some } \rho > 0\}$. Parashar and Choudhary [61] gave an extension of this sequence space and defined $X(\phi, (p_k)) = \{x = (x_k) \in \ell_0 : (\phi(\frac{|x_k|}{\rho})) \in X((p_k))\}$,
for $X \in \{\ell, \ell_\infty, c, c_0\}$. After that, Tripathy and Sarma [75] defined the