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**ON SOME TOPOLOGICAL APPROXIMATIONS in
INFORMATION SYSTEMS**

A Thesis

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TO

My Family

My Father, My Mother, My Husband,

My Brothers and My Sisters

With My Appreciation for Their Patience, Support and Love

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Introduction

Most real life situations need some sort of approximation to fit mathematical models. The beauty of using topology in approximation is achieved via obtaining approximation for qualitative concepts (i.e. subsets) without coding or using assumption. The aim of this Thesis is to apply near boundary notions in the approximation spaces. The basic notions of near approximations, near boundary regions and near accuracy are introduced and sufficiently illustrated. Near approximations considered as mathematical tools to modify the approximations of concepts.

A rough set is a formal approximation of a crisp set (i.e. conventional set) in terms of a pair of sets which gives the lower and the upper approximations of the original set. The lower and upper approximation sets themselves are crisp sets in the standard version of rough set theory [30, 32]. Rough set theory handled the uncertainty in general and the vagueness and imprecision in particular, since the main point of rough set theory is the boundary region which is the difference between the upper and the lower approximations. If the boundary region is empty, then it is a crisp set. In addition, many publications recently used rough sets to handle the incompleteness problems. Rough set theory is still one of the approaches to vagueness. Imprecision in this approach is expressed by a boundary region of a set.

Relations are used in the construction of topological structures in many fields such as dynamics [8], rough set theory and approximation space [30,32], biochemistry [44] and biology [45]. In fact, topology is branch of mathematics, whose concepts exist not only in almost all branches of mathematics, but also in much real life application. It should be noted that the generation of topology by relations and the representation of topological concepts via relations will narrow the gap between topology and its applications. Moreover, we obtain a quasi-discrete topology using a symmetric relation, which is considered as a generalization for the equivalence relation in [30, 32]. Rough set theory is a new approach to vagueness and uncertainty, and from a practical point of view, it is a new method of data analysis. Rough set theory has a wide variety of applications. It can be used for information preserving data reduction, representation of uncertain or imprecise knowledge, knowledge

discovery, concept classification, machine learning, data mining [47] economics [18], medical diagnosis [28] , and others [49].

A basic notion of rough set theory is the lower and upper approximation, or approximation operators [30, 32, 57], This theory can be developed in at least two different manners, the constructive and algebraic methods [58,62]. The constructive methods [36, 50], define rough set approximation operators using equivalence relations or their induced as abstract operators. There are several definitions of constructive methods, commonly known as the element based, granule based [37, 60], and subsystem based definitions [58]. Each of them offers a unique interpretation of the theory. They can be used to investigate the connections to other theories, and to generalize the basic theory in different directions. The element based definition establishes a connection between approximation operators and the necessity and the possibility operators of modal logic [62]. Under the granule based definition, one may view rough set theory as a concrete example of granular computing [59]. The subsystem based definition relates approximation operators to the interior and closure operators of topological spaces [53] , the closure operators of closure systems[62], and operators in other algebraic systems [41,63].

Classical definitions of lower and upper approximation [30, 32] were originally introduced with reference to indiscernibility relation which was assumed to be an equivalence relation (reflexive, symmetric, and transitive). It is quite interesting to extend these concepts to the case of more general relations. In particular, considering a similarity or tolerance relation, instead of an indiscernibility relation, is quite relevant [7, 19, 21, 27, 38, 42, 43, 48, and 64]. Such relations express weaker forms of indiscernibility and usually are not equivalence relations. While the reflexivity property seems quite necessary to express any form of indiscernibility or similarity, the two other properties may be relaxed.

One of the most powerful notions in system analysis is the concept of topological structures [12] and their generalizations. Many works have appeared recently for example in structural analysis [15], in chemistry [44], and physics [8]. The purpose of the present work is to put a starting point for the applications of abstract topological theory into fuzzy set theory, granular computing [61], rough set analysis and probability theory. Through this thesis, we shall integrate some ideas in terms of concepts in topology and uncertainty.

The classes of near open sets can be considered as rich sources for elementary concepts in information systems. These classes have been used extensively in abstract topological results. The purpose of this thesis is to spotlight on using these classes as tools for measuring the exactness of concepts. Basic notions of near rough near exact and near fuzzy sets are introduced. The topology induced by a binary relation is used to generalize the basic near rough concepts. We introduce near exactness and near roughness by applying near concepts to make more accuracy for definability of sets. We give a new definition for a membership function to find near interior, near boundary and near exterior points.

In this thesis, which consists of four chapters, we generalized both constructive and algebraic method for defining rough sets.

In chapter one: In this chapter, we introduce some basic concepts to be used through this thesis. These concepts are considered as a fundamental concept in topology, rough set theory.

In Chapter two: In the context of Chapter (2), we study the approximate space with general relations from topological view. We used some of near open sets to introduce new definitions and levels of the upper and lower approximations, and this leading to many new results. Also in light of these results we reached to different levels of the boundary regions, in addition we reached to a several levels of accuracy for the sets. Some of the results in Chapter (2) are published in

- ***Journal of the Advances and Applications in Mathematical Sciences.***

In chapter three: In this chapter we enlarged the number of possibility of definability by using the notions rough and semi rough sets in general topological spaces. The first section of this chapter presents a brief introduction to semi rough approximation and semi roughness, Section (2) we introduce semi rough membership function. Finally Section (3) describes the semi roughness bounds using properties of definability rough sets and investigates semi roughness bounds under different set operations for ordinary rough sets. Also we provides an example to show the derived bounds from operands roughness, and discusses the results. Some of the results in Chapter (3) are published in

- ***Journal of the Advances and Applications in Mathematical Sciences.***

In chapter four we generalized the subsystem based definition by using a topological structure generated from a general relation. We constructed new

approximations based on the well known topological notions of pre closures and pre interiors and studied their properties. The number of possible membership relations was enlarged and properties of these approximations were, also, discussed. Also we introduced new types of rough definability and undesirability, based on the notions of prelower and preupper approximations.

Chapter I

Introduction and Fundamental Notions

1.1 Fundamental notions of topological space

Topology is a branch of mathematics, whose concepts exist not only in almost all branches of mathematics, but also in many real life applications. Some of the basic concepts in topology which are useful for our study are given through this section. Also topology is an important and interesting area of mathematics, the study of which will not only introduce you to new concepts and theorems but also put into context old ones like continuous functions. It is so fundamental that its influence is evident in almost every other branch of mathematics. This makes the study of topology relevant to all who aspire to be mathematicians whether their first love is algebra, analysis, category theory, chaos, continuum mechanics, dynamics, geometry, industrial mathematics, mathematical biology, mathematical economics, mathematical finance, mathematical modeling, mathematical physics, mathematics of communications, number theory, numerical mathematics, operation research or statistics. Topological notions like compactness, connectedness and denseness are as basic to mathematicians of today as sets and functions were to those of last century [6,24,25]. For a long time, many individuals believed that abstract topological structures have limited application in the generalization of real line and complex plane or some connections to algebra and other branches of mathematics. And it seems that there is a big gap between these structures and real life applications. We noticed that in some situations, the concept of relation is used to get topologies that are used in important applications such as computing topologies[31], recombination spaces[5,22,53], and information granulation[59], which are used in biological sciences and some other fields of applications.

The aim of rough set theory is to give a description of the set of objects by logical, set-theoretical, topological etc. tools in terms of similarity relations and derived notions related by these relations. The description of the set of objects entails as well relationships and functional or near to functional dependencies among various similarity relations generated by various sets of objects. So, in this chapter we shall introduce some basic notions about topology and uncertainty theories. Section (1)

contains fundamental concepts of topological spaces. The purpose of Section (2) is to give an account on some uncertainty theories.

Definition 1.1.1[40]

A topological space (U, T) consists of a set U and a family $T \subseteq P(U)$ such that:

- 1- U & Φ belong to T .
- 2- If $B_1, B_2 \in T$ then $B_1 \cap B_2 \in T, \forall B_1, B_2$.
- 3- If $G_i \in T \Rightarrow \bigcup_{i \in I} G_i \in T, \forall i \in I$.

The family T is called a topology on U and the member of T are called open sets. The complement of an open set is a closed set. The family of closed sets is denoted by $T^c = \{X^c : X \in T\}$.

Definition 1.1.2[40]

Let (U, T) be a topological space. A family of sets $\beta \subseteq T$ is called a base for T if every non-empty open subset of U can be represented as a union of subfamily of β .

Proposition 1.1.1

Let β be a class of subsets of, $(U \neq \Phi, \beta \subset P(U))$ then β is a base for some Topology T if

- 1- $X = \bigcup \{B : B \in \beta\}$.
- 2- For all B, B^* in $\beta, p \in B \cap B^*, \exists B_p \in \beta \ni p \in B_p \subset B \cap B^*$.

Definition 1.1.3[13]

Let (U, T) be a topological space. A family $S \subseteq T$ is called a subbase for T iff the family of all finite intersections of S is a base for T .

Definition 1.1.4[13]

If (U, T) is a topological space and $X \subseteq U$, then:

- 1- $\bar{X} = \bigcap \{F \subseteq U : X \subseteq F \text{ \& } F \text{ is closed}\}$ is called the T – closure of X .

Evidently, $\bar{X}$ is the smallest closed subset of U which contains X . Note that X is closed if $\bar{X} = X$.

- 2- $X^\circ = \bigcup \{G \subseteq U : G \subseteq X \text{ \& } G \text{ is open}\}$, is called the T – interior of X . Evidently, X° is the union of all open subsets of U which containing in X . Note that X is open if $X = X^\circ$.

Definition 1.1.5[40]

If (U, T) is a topological space and $x \in U$, a neighborhood of x is a set N which contains an open set G containing x . Thus evidently N is a neighborhood of x iff $x \in N^\circ$.

Definition 1.1.6 [13]

Let (U, T) be a topological space and $X \subset U$, the boundary of X in U is the set $BN(X) = \overline{X} - X^\circ$ or $BN(X) = \overline{X} \cap \overline{X^c}$.

Definition 1.1.7 [18]

If (U, T) is a topological space, then the closure operator associated with it is defined as the function $c: P(U) \rightarrow P(U)$ such that $c(X) = \overline{X}$ for each $X \in P(U)$.

Definition 1.1.8 [18]

If (U, T) is a topological space, then the interior operator associated with it is defined as the function $i: P(U) \rightarrow P(U)$ such that $i(X) = X^\circ$ for each $X \in P(U)$.

Definition 1.1.9 [39]

A topology T on U is called a quasi-discrete topology if every open set is closed.

Example 1.1.1

Let (U, T) be a topological space, $U = \{a, b, c, d\}$ and $\beta = \{\{a\}, \{d\}, \{b, c\}\}$ is the base of T then T is a quasi-discrete topology.

Definition 1.1.10 [13]

A topology T on U is called an Alexandrov topology if the intersection of every family of open sets is also open. If T is Alexandrov topology on, the pair (U, T) is called Alexandrov space.

Proposition 1.1.2 [13]

If (U, T) is topological space, then the following assertions are equivalent

- 1- T is an Alexandrov topology.
- 2- Every point $x \in U$ has a smallest neighborhood.

Proposition 1.1.3 [13]

Let $U \neq \emptyset$, $S \subset P(U)$, S is a subbase for some topology T on $U \Leftrightarrow$

$\beta = \{B: B = \bigcap_{i=1}^n S_i, S_i \in S, i = 1, 2, \dots, n\}$ is a base for T .

Properties of closure of any set

If (U, T) is a topological space, $X, Y \subseteq U$, then :

- 1- $X \subset \overline{X}$.
- 2- $\overline{\emptyset} = \emptyset$.
- 3- $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$.
- 4- $\overline{X \cap Y} \subset \overline{X} \cap \overline{Y}$.
- 5- $\overline{\overline{X}} = \overline{X}$ s.t. $\overline{X} = \bigcap \{F: F \text{ closed}, X \subset F\}$.

Properties of interior of any set

If (U, T) is topological space, $X, Y \subseteq U$, then

- 1- $X^\circ \subset X$.
- 2- $U^\circ = U$.
- 3- $(X \cap Y)^\circ = X^\circ \cap Y^\circ$.
- 4- $(X \cup Y)^\circ \supset X^\circ \cup Y^\circ$.
- 5- $(X)^\circ = X^\circ$ s.t. $X^\circ = \bigcup \{G \subset U, G \in T, G \subset X\}$.

Proposition 1.1.4 [9]

Let (U, T) be a topological space then:

- 1- $U - \overline{X} = (U - X)^\circ$.
- 2- $U - X^\circ = \overline{U - X}$ for all $X \subseteq U$.

1.2 Rough set approximations

Rough set theory [30] proposes a new mathematical approach to imperfect knowledge, i.e. to vagueness (or imprecision). In this approach vagueness is expressed by region of a set.

Rough set concept can be defined by means of topological operations, interior and closure, called approximations.

Definition 1.2.1 [30, 32]

A knowledge base is system $K = (U, \mathcal{R})$ where $U \neq \emptyset$, and $\mathcal{R}$ be a family of equivalence relations over U .

Definition 1.2.2 [32]

Let U be a universe set such that $U \neq \emptyset$, $\mathcal{R}$ be a family of equivalences relation, $\mathcal{R} = \{R: R \text{ equivalence relation}\}$, $P \subseteq \mathcal{R}$ then $\bigcap P$ is call indiscernibility relation and denoted by $IND(P)$, i.e. $IND(P) = \bigcap P$.

Remark 1.2.1 [32]

By $IND(K)$ we denote the family of all equivalence relations defined in K , $IND(K) = \{IND(P): P \subseteq \mathcal{R}\}$.

Definition 1.2.3 [30, 32]

Suppose we given knowledge base $K = (U, \mathcal{R})$ with each subset $X \subseteq U$, and R , is an-equivalence relation, $R \in IND(K)$ we associate two sub sets: