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Oscillation Criteria of Second Order Forced Delay Dynamic Equations

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Contents

Summary

The study of dynamic equation on a time scales goes back to its founder Stefan Higler [24]. He aims to unify, extend and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time scale calculus. A time scale $\mathbb{T}$ is a nonempty closed subset of real numbers. When the time scale is the set of real numbers, the general results yield the results of ordinary differential equations. On the other hand, when the time scale is the set of integers, the same general results yield the results of difference equations. The new theory of the so-called "dynamic equation" is not only unify the theories of differential equations and difference equations, but also extends the classical cases to the so-called q-difference equations (when $\mathbb{T} = q^{\mathbb{N}_0} := \{q^t : t \in \mathbb{N}_0,\ q > 1\}$ or $\mathbb{T} = q^{\overline{\mathbb{Z}}} = q^{\mathbb{Z}} \cup \{0\}$) which have important applications in quantum theory (see[27]).

A delay differential equation (DDE) is an equation contains a function of a single variable (usually called time) and its derivatives. The derivative of the function at a certain time is given in terms of its value at earlier times. In the last two decades, there has been increasing interest in obtaining sufficient conditions for oscillation (nonoscillation) of the solutions of delay dynamic equations on time scales. So, we chose the title of the thesis " Oscillation Criteria of Second Order Forced Delay Dynamic Equations". We aim to use a class of Philos-type functions [38] and the generalized Riccati transformation in establishing some new oscillation criteria for the forced delay dynamic equations.

The thesis is devoted to the following:

1. Illustrate the new theory of Stefan Hilger through a general introduction to

the theory of dynamic equations on time scales,

2. Summarize some of the recent developments for the oscillation of second order forced delay differential equations and forced delay dynamic equations on time scales,

3. Establish some new sufficient conditions to ensure that all solutions of second order forced delay dynamic equations discussed in the thesis on unbounded time scales are oscillatory.

Chapter 1 contains the basic concepts of the theory of functional differential equations and some preliminary results of the theory of second order forced delay differential equations.

In Chapter 2, we give an introduction to the theory of dynamic equations on time scales, differentiation, integration, and various properties of the exponential function on arbitrary time scale. Also, we present the most important studies for the oscillation theory of second order forced delay dynamic equations on time scales.

In Chapter 3, we establish some new oscillation criteria for the second-order nonlinear functional dynamic equation with forced term

$$(r(t)x^{\Delta}(t))^{\Delta} + p(t)f(x(\tau(t))) = e(t),\ t \in \mathbb{T},\ t \geq t_0,$$

or

$$(r(t)x^{\Delta}(t))^{\Delta} - p(t)f(x(\tau(t))) = e(t),\ t \in \mathbb{T},\ t \geq t_0,$$

on a time scale $\mathbb{T}$ by using a class of Philos type functions. The following cases are taken into consideration:
(i) $p(t) > 0$, $\tau(t) \leq t(\geq t)$ and $\tau(t) \leq \sigma(t)(\geq \sigma(t))$.
(ii) $p(t)$ changes its sign, $\tau(t) \leq t(\geq t)$, $\tau : \mathbb{T} \to \mathbb{T}$ is a strictly increasing differentiable function and $\lim_{t\to\infty} \tau(t) = \infty$.
The obtained results are given in [4].

Chapter 4, presents some new oscillation criteria for the second-order forced nonlinear functional dynamic equations with damping term

$$(r(t)x^{\Delta}(t))^{\Delta} + q(t)x^{\Delta}(t) + p(t)f(x(\sigma(t))) = e(t), \tag{1}$$

and

$$(r(t)x^{\Delta}(t))^{\Delta} + q(\sigma(t))x^{\Delta}(t) + p(t)f(x(\tau(t))) = e(t), \tag{2}$$

on a time scale $\mathbb{T}$, where $r(t)$, $p(t)$, $q(t)$ and $e(t)$ are real-valued right-dense continuous functions with $p(t) < 0$ and no restrictions imposed on the forcing term $e(t)$ to satisfy Kartsatos condition. Our results improve and extend some results established by Sun and Wong [44] and also answer their question for the oscillation when $0 < \nu < 1$. These results are given in [6].

In Chapter 5, we use Riccati transformation technique to establish some new oscillation criteria for the second-order nonlinear forced dynamic equation with damping

$$(r(t)g(x^{\Delta}(t)))^{\Delta} + p(t)g(x^{\Delta}(t)) + q(t)f(x^{\sigma}(t)) = G(t, x^{\sigma}(t)),\ t \in \mathbb{T},\ t \geq t_0. \tag{3}$$

on time scale $\mathbb{T}$, where $r(t)$, $p(t)$ and $q(t)$ are real-valued right-dense continuous functions and no sign conditions imposed on these functions. Our results extend and improve some previous results established by Sun *et al.* [46]. These results are given in [5]

Chapter 1

Preliminaries

In this chapter, we give a survey of some previous studies related to the subject of the thesis. Also, we present some basic concepts of boundary and functional differential equations. Preliminary results that can be needed through our work are given. Later, we give some of the recent developments in oscillation theory of second order delay differential equations and investigate the existence of oscillatory solutions of second order delay differential equations.

1.1 Initial Value Problems

In this section, we give the definitions of ordinary and functional differential equations.

Definition 1.1.1 *An ordinary differential equation (ODE) is a relation that contains function of only one independent variable, and one or more of its derivatives with respect to the variable.*

Definition 1.1.2 *[42] A functional equation (FE) is an equation involves an unknown function for different argument values. The difference between the argument values of the unknown function and t in the FE are called argument deviations.*

Example 1.1.1 *The equations* $x(3t)+4t^3x(6t) = 4$*, and* $x(t) = e^t x(t+1)-[x(t-3)]^2$ *are examples of FEs.*

Remark [42]

If all argument deviations are constants (as in the second equation of the above example), then the FE is called a difference equation.

Combining definition 1.1.1 and definition 1.1.2, we obtain the definition of functional differential equation (FDE), or equivalently, differential equation with deviating argument as follows:

Definition 1.1.3 *[42] A functional differential equation is an equation contains an unknown function and some of its derivatives for different argument values. The order of a FDE is the order of the highest derivative of the unknown function. So, a FE may be regarded as a FDE of order zero.*

Definition 1.1.4 *The ordinary differential equation*

$$x^{'}(t) = f(t, x) \tag{1.1}$$

together with the equation

$$x(t_0) = x_0 \tag{1.2}$$

is called an initial value problem (Eq. (1.2) is called an initial condition).

It is well known that under certain assumptions on f the initial value problem (1.1) and (1.2) has the unique solution,

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))ds \qquad for\ t \geq t_0 \tag{1.3}$$

Definition 1.1.5 *The differential equation of the form*

$$x^{'}(t) = f(t, x(t), x(t-\tau)) \qquad with \qquad \tau > 0\ \ and\ \ t \geq t_0, \tag{1.4}$$

in which the right-hand side depends on the instantaneous position $x(t)$ and the position at τ units back $x(t-\tau)$, is called an ordinary differential equation with delay or a delay differential equation.

Whenever necessary, we shall consider the integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\tau))ds, \tag{1.5}$$

equivalent to (1.4). Eq. (1.4) with the initial condition

$$x(t) = \varphi(t) \qquad for\ all\ \ t \in E_{t_0}, \tag{1.6}$$

where φ is a known initial function on the initial set E_{t_0}, t_0 is an initial point is known as the initial value problem for a delay differential equation. Usually, it is assumed that $\varphi(t_0+0) = \varphi(t_0)$. By a one-sided derivative, we mean the derivative at one of the endpoint of an interval. Under general assumptions, the existence and uniqueness of the solution of the initial value problem (1.4) and (1.6) is given in Ladas [20]. The solution is sometimes denoted by $x(t,\varphi)$. In case of variable delay $\tau = \tau(t) > 0$, the initial set E_{t_0} has the form:

$$E_{t_0} = t_0 \cup \{t - \tau(t) : t - \tau(t) < t_0, t \geq t_0\}.$$

If it is required to determine the solution on the interval $[t_0, T]$, then the initial set is

$$E_{t_0 T} = t_0 \cup \{t - \tau(t) : t - \tau(t) < t_0, t \geq t_0 \leq T\}.$$

Example 1.1.2 *[1] for the equation*

$$x^{'}(t) = f(t, x(t), x(t - \cos^2 t)),$$

$t_0 = 0$, $E_0 = [-1, 0]$, *and the initial function φ must be given on the interval* $[-1, 0]$.

Definition 1.1.6 *A point x of a set S is called an isolated point of S if there exists a neighborhood of x contains no points of S other than x itself.*

Now, consider the differential equation of order n with l deviating arguments,

$$\begin{aligned} x^{(m_0)}(t) = f(t, x(t), ..., x^{(m_0-1)}(t), x(t-\tau_1(t)), ..., x^{(m_1-1)}(t-\tau_1(t)), ..., \\ x(t-\tau_l(t)), ..., x^{(m_l-1)}(t-\tau_l(t))), \end{aligned} \tag{1.7}$$

where the deviations $\tau_i(t) > 0$, and $n = \max\{\max_{1\leq i\leq l}(m_i - 1), m_0\}$. In order to formulate the initial value problem for (1.7), we assume that t_0 is the given initial point and for each deviation $\tau_i(t)$, the initial set $E_{t_0}^{(i)}$ is given by

$$E_{t_0}^{(i)} = t_0 \cup \{t - \tau_i(t) : t - \tau_i(t) < t_0, t \geq t_0\}.$$

We denote $E_{t_0} = \cup_{i=1}^{l} E_{t_0}^{(i)}$. On E_{t_0}, continuous functions φ_k, $k = 0, 1, ..., \lambda$, must be given with $\lambda = \max_{1\leq i\leq l}(m_i - 1)$. In applications, we often (not generally) consider the initial conditions,

$$\varphi_k(t) = \varphi_0^{(k)}(t) \qquad for\ k = 0, 1, ..., \lambda.$$

For the nth order differential equation, there should be given initial values $x_0^{(k)}$, $k = 0, 1, 2, ..., n-1$. Now let $x_0^{(k)} = \varphi_k(t_0)$, $k = 0, 1, 2, ..., \lambda$. If $\lambda < n-1$, then the numbers $x_0^{(\lambda+1)}, ..., x_0^{(n-1)}$ must be given. Also if the point t_0 is an isolated point of E_{t_0}, then $x_0^{(0)}, ..., x_0^{(n)}$ must be given. For (1.7), the basic initial value problem consists of the determination of an $(n-1)$ times differentiable function x that satisfies (1.7) for $t > t_0$,

$$x^{(k)}(t_0 + 0) = x_0^{(k)}, \qquad k = 0, 1, 2, ..., n-1$$

and

$$x^{(k)}(t - \tau_i(t)) = \varphi_k(t - \tau_i(t)) \qquad for\ \ t - \tau_i(t) < t_0,$$

where $k = 0, 1, 2, ..., \lambda$ and $i = 0, 1, 2, ..., l$. At the point $t_0 + (k-1)\tau$ the derivative $x^{(k)}(t)$ is in general discontinuous, but the derivatives of lower order are continuous.

Example 1.1.3 *[1] Consider*

$$x^{''}(t) = f(t, x(t), x^{'}(t), x(t - \cos^2(t)), x(\frac{t}{2})). \tag{1.8}$$

In this case, we have $n = 2$, $l = 2$ *and* $\lambda = 0$. *For* $t_0 = 0$, *the initial sets are* $E_0^1 = [-1, 0]$, *and* $E_0^{(2)} = 0$. *Hence* $E_0 = [-1, 0]$. *On* E_0, *the initial function* φ_0 *is given. Also, the initial values* $x_0^{(0)} = \varphi_0(0)$, *and* $x_0^{(1)} = \varphi_1(0)$ *are given numbers.*

For (1.7) a classification method was proposed by Kamenskiǐ [28]. Let $\beta = m_0 - \lambda$. *If* $\beta > 0$, *(1.7) is called an equation with retarded arguments or with delay. If* $\beta = 0$, *(1.7) is called an equation of neutral type. If* $\beta < 0$, *it is called an equation of advanced type.*

Example 1.1.4 *[1] The equations*

$$x^{'}(t) + a(t)x(t - \tau) = 0 \qquad with \qquad \tau > 0,$$

$$x^{'}(t) + a(t)x(t + \tau) = 0 \qquad with \qquad \tau > 0,$$

and

$$x^{'}(t) + a(t)x(t) + b(t)x^{'}(t - \tau) = 0 \qquad with \qquad \tau > 0,$$

are of retarded type ($\beta = 1$), *advanced type* ($\beta = -1$), *and neutral type* ($\beta = 0$), *respectively.*

The above classification is incomplete. For example, the equation

$$x^{'}(t) + ax(t - \tau) + ax(t + \sigma) = 0, \qquad \tau > 0,\ \sigma > 0 \tag{1.9}$$

does not belong to the above three classes.
Sometimes Eq. (1.9) is called an equation of mixed type.

1.2 Introduction to Delay-Differential Equations

Delay-differential equations (DDEs) form an important class of dynamical systems. They are often arise in either nature or technological control problems. In these systems, a controller monitors the state of the system, and makes an adjustment to the system based on its observations. Since these adjustments can never be made instantaneously, a delay arises between the observation and the control action.

In mathematics, delay differential equations (DDEs) are considered as differential equations in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times.

There are different kinds of delay-differential equations, such kinds are:

1- Continuous delay:

$$\frac{d}{dt}x(t) = f\big(t, x(t), \int_{-\infty}^{0} x(t+\tau)d\mu(\tau)\big).$$

2- Discrete delay:

$$\frac{d}{dt}x(t) = f(t, x(t), x(t-\tau_1), x(t-\tau_2), ..., x(t-\tau_m)),\ for\ \tau_1 > ... > \tau_m \geq 0.$$

3- Linear with discrete delays:

$$\frac{d}{dt}x(t) = A_0x(t) + A_1x(t-\tau_1) + ... + A_mx(t-\tau_m),$$

where $A_0, ..., A_m \in \mathbb{R}^{n\times n}$.

4- Pantograph equation:

$$\frac{d}{dt}x(t) = ax(t) + bx(\lambda t),$$

where a, b and λ are constants and $0 < \lambda < 1$.

In the following, we study equations with discrete delays.

In order to solve a delay equation, we need to consider the earlier values of x at each time step. Therefore, we need to specify an initial function which gives the behavior of the system prior to time 0 (assuming that we start at $t = 0$). This function has to cover a period of time as long as the longest delay. For example, the initial function of the class of equations with a single delay,

$$x^{'}(t) = f(x(t), x(t-\tau)),$$

must be a function $x(t)$ defined on the interval $[-\tau, 0]$.

1.3 Application

Most of the differential equation models in dynamics have been derived starting from the simple equation:

$$\frac{dx(t)}{dt} = x(t),$$

where $x(t)$ denotes the density of a population (or biomass) of a single species at time t.
To derive model equations with delay in production and destruction in the following balance equation, assuming there is no immigration or emigration

$$\frac{dx(t)}{dt} = birth\ rate - death\ rate,$$

For instance, if we consider a population of adult flies, then the production or recruitment of adult flies at time t depends on the population of adult at time $t - \tau$, where τ is the time required for the larvae to become adult. If the birth and death rates are governed by density dependent factors, then we have

$$\frac{dx(t)}{dt} = a(x(t-\tau)) - b(x(t)),$$