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Applications of Variational Principles, Similarity and Travelling Wave Solutions to Nonlinear Evolution Type Equations.

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Abstract

This thesis consists of five chapters, together with an Arabic and English summaries as follows:

Chapter (I): Comprise an introduction containing a brief survey and the development of the available literature relevant to the methods used in chapters II-V.

Chapter (II): Here in we apply the variational principles method for solving the variable coefficients nonlinear Schrödinger equation, the modified Zakharov-Kuzentsov equation and combined KdV-mKdV equation. We discussed the existence of the Lagrangian and the invariant variational principles for equations under study. By using Noether´s theorem we have obtained conservation laws. The motivation for this chapter is to enlarge the variation technique to obtain the exact solutions of those equations.

Chapter (III): In this chapter KdV type equations with variable coefficient (KdV, mKdV and Burger KdV) which appear in arterial mechanics have been analyzed via symmetry method. By using the infinitesimal symmetries, there are four basic fields determined for the first equation while in the second and third equations only three basic fields are obtained. These fields help us to reduce the nonlinear partial differential equations into nonlinear ordinary differential equations. The search for solutions to those reduced ordinary equations, corresponding to the equation under consideration, has yielded certain classes of exact solutions for KdV type equations with variable coefficient. Some of those solutions recover other solutions obtained before in literature and others are new.

Chapter (IV): We apply the Bäcklund transformations on the combined KdV-Burgers equation, the coupled KdV equations and the new generalized Zakharov-Kuznetsov equation with variable coefficients. This leads to solitons and other type solutions.

Chapter (V): In this chapter we have applied two integral methods, the first and the direct integral methods to find many types of solutions for the following equations: the higher-order nonlinear Schrödinger equation, the (2+1)-dimensional Davey-Stewartson system, and the (2+1)-dimensional dispersive long wave equation.

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CHAPTER I

INTRODUCTION AND SURVEY OF LITERATURE

1.1 Introduction

An evolution equation usually means a partial differential equation with one of the independent variables being the time t. There are many nonlinear evolution equations arising from physics, mechanics, biology, chemistry, material science and plasma physics etc. in many different fields of science and engineering, it is very important to obtain exact solutions for nonlinear evolution equations. In recent years, the investigation of those exact solutions plays an important role in the study of nonlinear physical phenomena. From this importance we confine our attention in this thesis to find exact solutions for nonlinear evolution equations using various methods.

The first method is the variational principles method for solving nonlinear partial differential equations which appear in a wide variety of modelling physical phenomena and applications is very important in physical field becouse itś allow us to obtain the Lagrangian, the generators of the infinitesimal Lie groups, and for writing down the conservation laws via Noether's theorem see (Logan [33], Bhutani and Mital [5], Bhutani and Sharma [4], Bulman and Cole [3], Olver [52], Noether [48] and Khater et al. [32]). The motivation for the present work is to enlarge the technique for obtaining exact solutions.

The second method is the symmetry method given by Steinberg [56]. In his report he discussed a method of finding explicit solutions of both linear and nonlinear partial differential equations (NLPDEs). The classical version of his method is usually referred to as similarity method.

However, Steinberg's method is more algorithmic and more general than the classical Lie method (Bluman and Kumei [7], Olver [52]). In the case of linear equations this method bears a close relationship to the method of separation of variables and in fact, this method produces a large supply of separable solutions for linear equations.

One of the most important thing for Steinberg's symmetry method is that it is a computational procedure that can be used by any person familiar with differential equations but not familiar with Lie group theory. Although, the person who is familiar with Lie group theory will find that the elementary methods that used group theory to find similarity solutions are not powerful enough to give a complete analysis of the heat and Burger equations as the symmetry method of Steinberg gave in his report. Bhutani et al [8] enlarged this technique to deal with systems of partial differential equations then the technique has found an important place in the literature of group theoretic methods see (Moussa et al. [45-47], Moussa [41], El Shikh [16] and El-Gazzy [17]).

The third method is Bäcklund transformations of PDEs which plays an important role in solitary theory. In 1983, Weiss, Tabor and Carnevale generlized the Painlevé property of ordinary differential equations (ODEs) and presented WTC method that directly test the Painlevé property of PDEs. By the truncation of Painléve expansion at the constant level term, we can obtain Bäcklund transformation of nonlinear PDEs (Lou [35-36], Weiss [67], El-Kalaawy [14]). Another powerful tool is the homogeneous balance (HB) method it is used to find solitary wave solutions of nonlinear (PDE) (Fan [18], Wang [68-69], Feng and Fan [21]).In Fan [18] the HB improved to obtain more other kinds of exact solutions. Fan [19, 20] extended the HB to search for Bäcklund transformation of nonlinear PDEs. He found that there exists a close connections among the HB method and WTC method. Recently, Moussa et al. [44] have enlarged this connection between the WTC method and HB to obtain Bäcklund transformation for variable coefficients systems.

The fourth method is the first integral method. It is first proposed by Feng [23-24] to solve the Burgers-KdV equation. This method based on the ring theory of commutative algebra. Also, this useful method has been widely used by many researchers, such as in (Raslan [55],Abbasbandy [2], Hosseini et al. [29], Lu et al. [40], Tascan [64], Taghizadeh [66]) and the references therein.

Finally we have used an important method , its the generalized tanh method. The pioneer work of Malfiet [43] introduced this powerful hyperbolic tangent (tanh) tanh method for reliable treatement of the nonlinear wave equations. The usful tanh method is widely used by many such as in (Fan and Hong [22], Fan and Zhang [18], Yan [72]) and by references therein. The method introduces a unify method that one can find exact as well as approximate solutions in a straightforward and systematic way (El-Wakil et al [15]). The tanh method has been subjected to many modifications that mainly depend on the Riccati equation or the solutions of well-known equations and proposing a generalized ansatz. The standard tanh method and the proposed modifications all depend on the balance method, where the linear terms of highest order are balanced with the highest order nonlinear terms of the reduced equation. Now, we going to introduce those methods in details as follows

1.2 Variational principles method

We study existence and formulation of invaiant variational principles as follows:

The consistency conditions for the existence of a functional integral and the method for writing it down whenever it exists for a system of NLPDEs are summarized as follows:

For the system of two NLPDEs of second order in the form

$$N(u,v)=0,\ \ M(u,v)=0, \tag{1.1}$$

where $u=u(x,t)$ and $v=v(x,t)$. Following Atherton and Homsy [1], Bhutani and

Sharma [4], and Tonti [60-61], the consistency conditions are expressed as follows:

$$\frac{\partial N}{\partial u_x} = \frac{\partial}{\partial x}\left(\frac{\partial N}{\partial u_{xx}}\right) + \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial N}{\partial u_{xt}}\right), \tag{1.2}$$

$$\frac{\partial N}{\partial u_t} = \frac{\partial}{\partial t}\left(\frac{\partial N}{\partial u_{tt}}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial N}{\partial u_{xt}}\right), \tag{1.3}$$

$$\begin{aligned}\frac{\partial M}{\partial u} &= \frac{\partial N}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial N}{\partial v_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial N}{\partial v_t}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial N}{\partial v_{xx}}\right) \\ &+ \frac{\partial^2}{\partial x \partial t}\left(\frac{\partial N}{\partial v_{xt}}\right) + \frac{\partial^2}{\partial t^2}\left(\frac{\partial N}{\partial v_{tt}}\right),\end{aligned} \tag{1.4}$$

$$\frac{\partial M}{\partial u_x} = -\frac{\partial N}{\partial v_x} + 2\frac{\partial}{\partial x}\left(\frac{\partial N}{\partial v_{xx}}\right) + \frac{\partial}{\partial t}\left(\frac{\partial N}{\partial v_{xt}}\right), \tag{1.5}$$

$$\frac{\partial M}{\partial u_t} = -\frac{\partial N}{\partial v_t} + 2\frac{\partial}{\partial t}\left(\frac{\partial N}{\partial v_{tt}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial N}{\partial v_{xt}}\right), \tag{1.6}$$

$$\frac{\partial M}{\partial u_{xx}} = \frac{\partial N}{\partial v_{xx}}, \tag{1.7}$$

$$\frac{\partial M}{\partial u_{xt}} = \frac{\partial N}{\partial v_{xt}}, \tag{1.8}$$

$$\frac{\partial M}{\partial u_{tt}} = \frac{\partial N}{\partial v_{tt}}, \tag{1.9}$$

$$\frac{\partial M}{\partial v_x} = \frac{\partial}{\partial x}\left(\frac{\partial M}{\partial v_{xx}}\right) + \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial M}{\partial v_{xt}}\right), \tag{1.10}$$

$$\frac{\partial M}{\partial v_t} = \frac{\partial}{\partial t}\left(\frac{\partial M}{\partial v_{tt}}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial M}{\partial v_{xt}}\right), \tag{1.11}$$

Furthermore, if the system of Eq. (1.1) satisfies the above conditions,then a functional integral $J(u, v)$ can be written down using the formula given by Tonti [74] as:

$$J(u, v) = \int_\Omega u \left[\int_0^1 N(\lambda u, \lambda v) d\lambda\right] d\Omega + \int_\Omega v \left[\int_0^1 M(\lambda u, \lambda v) d\lambda\right] d\Omega \tag{1.12}$$

or through its equivalent statement

$$\delta J(u, v) = \int_\Omega N(u, v)\delta u d\Omega + \int_\Omega M(u, v)\delta v d\Omega \tag{1.13}$$

where, in Eqs. (1.12) and (1.13), $\int_\Omega d\Omega$ represents the integration over the domain Ω and $\int_0^1 d\lambda$ represents the integration over the scalar variable λ.

Now, we use Noetherś theorem to get conservation laws as follows:

The fundamental functional integral (Khater et al [32]) corresponding to the second order variational problems can be written as

$$J(u) = \int_{\Omega} L(x, u(x), \partial u(x), \partial^2 u(x)) dx^1 dx^m \tag{1.14}$$

where $x = (x^1, ..., x^m)$, $u(x) = (u^1(x), ..., u^n(x)) \in C_n^4(\Omega)$ is the set of all continuous functions on Ω whose fourth order partial derivatives are continuous, and $\partial u(x)$ and $\partial^2 u(x)$ denote the collection of the first and second partial derivatives

$$u_{\alpha}^{\cdot k} = \frac{\partial u^k}{\partial x^{\alpha}}, \qquad u_{\alpha\beta}^{\cdot\cdot k} = \frac{\partial^2 u^k}{\partial x^{\beta} \partial x^{\alpha}}$$

respectively and a r-parameter family of transformations

$$\begin{aligned} x^{\cdot\cdot\alpha} &= \Phi^{\alpha}(x, u, \epsilon) \\ u^{\cdot\cdot k} &= \Psi^{k}(x, u, \epsilon) \qquad (\alpha, \beta = 1, ..., m;\ k = 1, ..., n) \end{aligned} \tag{1.15}$$

The following theorems are of fundamental importance.

Theorem 1: If the fundamental funtional integral defined by Eq. (1.14) is invariant under the r-parameter family of transformation (1.15), then the Lagrangian L and its derivatives satisfy the r-identities:

for $s = 1, ..., r$, where

$$\begin{aligned} &\frac{\partial L}{\partial x^{\alpha}} \tau_s^{\alpha} + \frac{\partial L}{\partial u^k} \eta_s^k + \frac{\partial L}{\partial u_{\alpha}^{\cdot k}} \left(\frac{d\eta_s^k}{dx^{\alpha}} - u_v^{\cdot k} \frac{d\tau_s^v}{dx^{\alpha}}\right) \\ &+ \frac{\partial L}{\partial u_{\alpha\beta}^{\cdot\cdot k}} \left(\frac{d^2 \eta_s^k}{dx^{\alpha} dx^{\beta}} - u_{\beta v}^{\cdot\cdot k} \frac{d\tau_s^v}{dx^{\alpha}} - u_{v\alpha}^{\cdot\cdot k} \frac{d\tau_s^v}{dx^{\beta}} - u_v^{\cdot k} \frac{d^2 \tau_s^v}{dx^{\alpha} dx^{\beta}} \right) \\ &+ L \frac{d\tau_s^{\alpha}}{dx^{\alpha}} = 0, \end{aligned} \tag{1.16}$$

for $s = 1, ..., r$, where

$$\tau_s^\alpha(x,u) = \left.\frac{\partial \Phi^\alpha}{\partial \epsilon^s}\right|_{\in=0}, \quad \eta_s^k(x,u) \equiv \left.\frac{\partial \Psi^k}{\partial \epsilon^s}\right|_{\in=0}$$

Theorem 2 (Noethers̀ identity): under the hypothesis of theorem 1, the following r-conservation laws hold true

$$\frac{d}{dx^\alpha}\left[L\tau_s^\alpha + \left\{\frac{\partial L}{\partial u_\alpha^{\cdot k}} - \frac{d}{dx^\beta}\left(\frac{\partial L}{\partial u_{\alpha\beta}^{\cdot\cdot k}}\right)\right\} C_s^k + \frac{\partial L}{\partial u_{\alpha\beta}^{\cdot\cdot k}}\frac{dC_s^k}{dx^\beta}\right] = 0. \tag{1.17}$$

for $s = 1, ..., r$, where

$$c_s^k = \eta_s^k - u_\alpha^{\cdot k}\tau_s^\alpha$$

Finally, we apply a new technique to transform the partial differential equation under study to ordinary differential equation (ODE), where by solving it we could obtain exact solutions

1.3 Symmetry method

We briefly outlined Steinberg's [69] similarity method of finding explicit solutions of both linear and nonlinear partial differential equations . The method based on finding the symmetries of the differential equation as follows:
Suppose that the differential operator L can be written in the form

$$L(u) = \frac{\partial^p u}{\partial t^p} - H(u), \tag{1.18}$$

where $u = u(t,x)$ and H may depend on t, x, u and any derivative of u as long the derivative of u dose not contain more than $p-1$, t derivatives. Consider the symmetry operator (called infinitesimal symmetry) which is quasi-linear partial differential operator of first-order, in the form

$$S(u) = A(t,x,u)\frac{\partial u}{\partial t} + \sum_{i=1}^{n} B_i(t,x,u)\frac{\partial u}{\partial x_i} + C(t,x,u). \tag{1.19}$$

Define the Frèchet derivative of $L(u)$ by

$$F(L,u,v) = \frac{d}{d\varepsilon} L(u+\varepsilon v)|_{\varepsilon=0}. \tag{1.20}$$

With these definitions in mind we need to follow the following steps:

(i) Compute $F(L,u,v)$.

(ii) Compute $F(L,u,S(u))$.

(iii) Substitute $H(u)$ for $(\frac{\partial^p u}{\partial t^p})$ in $F(L,u,S(u))$.

(iv) Set this expression to zero and perform a polynomial expansion.

(v) Solve the resulting partial differential equations. Once this system of partial differential equations is solved for the coefficients of $S(u)$, equation under study can be used to obtain the functional form of the solutions.

1.4 Bäcklund transformation connected with the improved homogeneous balance (HB) method

Let us simply describe the main steps of this method as follows (Fan [20])

$$u_t = K(u, u_x, u_{xx}, ..). \tag{1.21}$$

According to the idea of HB method, we suppose that Bäcklund transformation of (1.21) takes the form

$$u(x,t) = \frac{\partial^\alpha f(w)}{\partial x^\alpha} + u_0, \tag{1.22}$$

where $f = f(w)$ and $w = w(x,t)$ are undetermined functions and u, u_0 are two solutions of Eq. (1.21). α can be determined by balancing between the linear term of u with the highest order nonlinear derivative term of u.

Substituting Eq. (1.22) into Eq. (1.21), putting all term of highest degree of w_x together and setting its coefficient to zero leads to an ODE, from which $f(w)$ is