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Fuzzy Structures of Some Algebras

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

(وقل رب زدني علماً)

"صدق الله العظيم"

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Abstract

Fuzzy Structures of Some Algebras

This thesis has been mainly divided into five chapters.

Chapter 1 highlights some basic definitions and results available in the standard literature. In this chapter we collect all the necessary preliminaries which will be useful in our discussions in the main text of the thesis.

Chapter 2 consists of two parts: In part (I) the notion of interval-valued fuzzy medial-ideals (briefly i-v fuzzy medial-ideal) in BCI-algebras is introduced. Several theorems are stated and proved. The image and pre-image of interval-valued fuzzy medial-ideals are defined and how the homomorphic images and pre-images of interval-valued fuzzy medial-ideals become interval-valued fuzzy medial - ideals in BCI-algebras is studied as well.

In (II) part, we introduce the notion of doubt intuitionistic fuzzy medial ideals, doubt intuitionistic fuzzy magnified translation medial ideals in BCI-algebras and investigate some interesting results. Moreover, some algorithms for medial ideals, fuzzy set and doubt intuitionistic fuzzy medial ideals have been constructed.

Chapter 3 consists of two parts: In first part, we introduce the concept of cubic medial-ideal and investigate several properties .Also, we give relations between cubic medial-ideal and cubic BCI-ideal .The image and the pre-image of cubic medial-ideal under homomorphism of BCI-algebras are defined and how the image and

the pre-image of cubic medial-ideal under homomorphism of BCI-algebras become cubic medial-ideal are studied. Moreover, the Cartesian product of cubic medial-ideal in Cartesian product BCI-algebras is given.

In the second part, we introduce the concept of bipolar fuzzy medial-ideal and investigate several properties. Also, we give relations between bipolar fuzzy medial-ideal and bipolar BCI-ideal. The image and the pre-image of bipolar fuzzy medial-ideal under homomorphism of BCI-algebras are defined and how the image and the pre-image of bipolar fuzzy medial-ideal under homomorphism of BCI-algebras become bipolar fuzzy medial-ideal are studied. Moreover, the Cartesian product of bipolar fuzzy medial-ideal in Cartesian product BCI-algebras is given.

Chapter 4 consists of two parts: In first part, the intuitionistic (T,S)-fuzzification of the concept medial ideals are considered. The image (pre-image) of (T,S)-fuzzy medial ideals under homomorphism of BCI-algebras are defined and some of their properties are investigated. The notion of product of intuitionistic (T,S)-fuzzy medial ideals in BCI-algebras are introduced and some related properties investigated.

In the second part, the generalization of notions intuitionistic (T,S)- fuzzy medial sub-algebras and intuitionistic fuzzy medial ideals in BCI-algebras by using t-norm T and s-norm S are given .Moreover some interesting results are investigated .

Chapter 5 consists of two parts: In first part, we introduce the concept of bipolar (T,S)-fuzzy medial-ideal and investigate several properties. Also, we give relations between bipolar (T,S)-fuzzy medial-ideal and bipolar (T,S)-fuzzy BCI-ideal. The image and the pre-image of bipolar (T,S)-fuzzy medial-ideal under homomorphism of BCI-algebras are defined and how the image and the pre-image of bipolar (T,S)-fuzzy medial-ideal under homomorphism of BCI-algebras become bipolar (T,S)-fuzzy medial-ideal are studied. Moreover, the Cartesian product of bipolar (T,S)-fuzzy medial-ideal in Cartesian product BCI-algebras is given.

In the second part, we consider the interval-valued intuitionistic $(\tilde{T}, \tilde{S})$ -fuzzification of the concept medial ideals. The image (pre-image) of interval-valued intuitionistic $(\tilde{T}, \tilde{S})$ -fuzzy medial ideals under homomorphism of BCI-algebras are defined and investigated some of there properties. We introduce the notion of product of interval-valued intuitionistic $(\tilde{T}, \tilde{S})$ -fuzzy medial ideals in BCI-algebras, and investigate some related properties.

Key words: BCI-algebras, medial ideals, fuzzy medial ideals, (T,S) norm, interval-valued fuzzy medial-ideals, doubt intuitionistic fuzzy medial ideals, cubic medial-ideals, bipolar fuzzy medial-ideals.

Summary

The notion of BCK-algebras was proposed by Iami and Iseki [16,17]. In 1966 Iseki [13,15] introduced the notion of a BCI-algebra, which is a generalization of BCK-algebra. Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK / BCI-algebras and their relationship with other structures including lattices and Boolean algebras. There is a great deal of literature which has been produced on the theory of BCK/BCI-algebras, in particular, emphasis seems to have been put on the ideal theory of BCK/BCI-algebras [13,26,42,44,45,52,58]. For the general development of BCK/BCI-algebras the ideal theory plays an important role. The concept of fuzzy sets was first introduced by Zadeh [53]. From that time, the theory of fuzzy sets which has been developed in many directions and found applications in a wide variety of fields [1,2,7, 8 ,9 ,10,11 ,13,18, 22,23,40,41,49,56]. In 1991, Xi [49] applied the concept of fuzzy sets to BCI, BCK, MV-algebras. The ideal theory and its fuzzification play an important role.

In [42] J. Meng and Y.B. Jun studied medial BCI-algebras. In [47] S.M. Mostafa, Y.B. Jun and A. El-Menshawy introduce the notion of medial ideals in BCI-algebras, they state the fuzzification of medial ideals and investigate its properties. There are several kinds of fuzzy sets extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval valued fuzzy sets, vague sets etc. Biswas in [9] gave the idea of anti fuzzy subgroups. Zadeh [53] made an extension of the concept of fuzzy set by an interval-valued fuzzy set (i.e., a fuzzy set with an interval valued membership function).

In [10], Biswas defined interval valued fuzzy subgroups and investigated some elementary properties. Jun [25] defined a doubt fuzzy sub-algebra, doubt fuzzy ideal, in BCI/ BCK -algebras and got some results about it. The idea of “intuitionistic fuzzy set” was first published by Atanassov [4,5] as a generalization of the notion of fuzzy sets. After that many researchers consider the Fuzzifications of ideals and sub-algebras in BCK/BCI-algebras. [18,46] introduced the notion of intuitionistic fuzzy (medial)- ideals and investigated some simple but elegant results. [28,39] discussed fuzzy translations, (normalized, maximal) fuzzy extensions and fuzzy multiplications of fuzzy sub-algebras in BCK/BCI-algebras and introduced the relations among fuzzy translations, (normalized, maximal) fuzzy extensions and fuzzy multiplications.

In [7,25,56,57], the authors have studied doubt (intuitionistic) fuzzy sub-algebras, doubt (intuitionistic) fuzzy ideals in BCK=BCI-algebras and introduced the relations among doubt (intuitionistic) fuzzy ideals and doubt (intuitionistic) fuzzy (H)-ideals [29,30,31,32,33,50] introduced the notion of cubic sub-algebras/ideals in BCK/BCI/KU-algebras, and then they investigated several properties. They discussed the relationship between a cubic sub-algebra and a cubic ideal. Also, they provided characterizations of a cubic sub-algebra/ideal and considered a method to make a new cubic sub-algebra from an old one.

On the other hand, triangular norm is a powerful tool in the theory research and application development of fuzzy sets see [24,34,35,40]. On the basis of the definition of the intuitionistic fuzzy

groups, Li [40] generalized the operators “ $\wedge$ ” and “ $\vee$ ” to T-norm and S-norm and defined the intuitionistic fuzzy groups of (T-S) - norms as a generalization of the notion of fuzzy set. K. H. Kim [34,35]. Using t-norm T and s-norm S, they introduced the notion of intuitionistic (T;S)-normed fuzzy sub-algebra in BCK/BCI/-algebra, Γ -rings and some related properties are investigated.

Jun et al. [18] considered the intuitionistic fuzzification of the concept of sub-algebras and ideals in BCK-algebras, and investigated some of their properties. They introduced the notion of equivalence relations on the family of all intuitionistic fuzzy ideals of a BCK-algebra and investigated some related properties. [3,19,20,21,36,37,38,43] introduced an extension of fuzzy sets named bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0,1]$ to $[-1,1]$.

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interval-valued intuitionistic $(\tilde{T}, \tilde{S})$ -fuzzy medial ideals in BCI-algebras, and investigate some related properties.

Key words: BCI-algebras, medial ideals, fuzzy medial ideals, (T, S) norm, interval-valued fuzzy medial-ideals, doubt intuitionistic fuzzy medial ideals, cubic medial-ideals, bipolar fuzzy medial-ideals.

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