Faculty of Education
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Oscillation of Second Order Dynamic Equations with Mixed Arguments on Time Scales

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Contents

Summary

Studing the dynamic equations on time scales was introduced by Stefan Hilger [28]. It is a new area of still fairly theoretical exploration in mathematics. Motivating the subject is a notion that dynamic equations on time scales can build bridges between continuous and discrete mathematics. Further, studying time scales lead to several important applications, e.g., insect population models, neural networks, and heat transfer. A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers. When the time scale equals the set of real numbers, the obtained results yield results of ordinary differential equations, while when the time scale equals the set of integers, the obtained results yield results of difference equations. The new theory of the so - called "dynamic equation" is not only unify the theories of differential and difference equations, but also extends the classical cases to the so - called q - difference equations (when $\mathbb{T} = q^{\mathbb{N}_0} := \{q^t : t \in \mathbb{N}_0, q > 1\}$ or $\mathbb{T} = q^{\overline{\mathbb{Z}}} = q^{\mathbb{Z}} \cup \{0\}$) which have important applications in quantum theory (see [31]).

A neutral differential equation with deviating arguments is a differential equation in which the highest order derivative of the unknown function appears with and without deviating arguments. In recent years, there has been an increasing interest in studying oscillation and nonoscillation of solutions of neutral dynamic equations on time scales which seek to harmonize the oscillation of continuous and discrete mathematics, however the study was restricted to specific equations under certain conditions. The oscillation conditions of their equations are not applicable when these conditions change. For this reason we aim to generalize these equations. So we select the title of our thesis to be "Oscillation of Second Order Dynamic Equations with Mixed Arguments on Time Scales" using the generalized Riccati transformation, the inequality technique and the generalized exponential function in establishing some new oscillation criteria for second order neutral dynamic equations with mixed arguments on time scales.

This thesis is devoted to

1. Illustrate the new theory of Stefan Hilger by giving an introduction to the theory of dynamic equations on time scales,
2. Summarize some of the recent developments in oscillation of second order neutral differential equations, oscillation of second order nonlinear neutral dynamic equations and oscillation of second order dynamic equations with damping on time scales.
3. Establish new sufficient conditions to ensure that all solutions of second order nonlinear neutral dynamic equations with mixed arguments and second order nonlinear mixed neutral dynamic equations with nonpositive neutral term on time scales are oscillatory or almost oscillatory or tend to zero, so that the obtained results are more generalized than those obtained in previous studies.
4. Trying to find some new oscillation criteria for second-order mixed nonlinear neutral dynamic equations with damping on time scales.
5. Give some examples to illustrate the importance of our results.

This thesis consists of five chapters :-

Chapter 1, is an introductory chapter that contains the basic concepts of studying the oscillation of solutions for functional differential equations as well as some previous results in studying the oscillation of second order neutral differential equations.

In Chapter 2, we give an introduction to the theory of dynamic equations on time scales, differentiation and integration, and some examples of time scales. Moreover, we present various properties of generalized exponential function on time scales. Additionally, some previous studies for the oscillation theory of second order neutral dynamic equations and second order dynamic equations with damping on time scales are presented.

In Chapter 3, we establish some new oscillation criteria for the second-order nonlinear neutral dynamic equation with mixed arguments on a time scale $\mathbb{T}$

$$\big(r(t)[\big(x(t)+p_1(t)x(\eta_1(t))+p_2(t)x(\eta_2(t))\big)^{\Delta}]^{\gamma}\big)^{\Delta}+f(t,x(\tau_1(t)))+g(t,x(\tau_2(t)))=0,$$

The results of this chapter generalize and extend the results of Tao Ji et al. [29], and published in: **Journal of Basic and Applied Research International 17(1)(2016) 49-66.** [5].

In Chapter 4, we present some new oscillation results for the second-order nonlinear mixed neutral dynamic equation with non positive neutral term on a time scale $\mathbb{T}$

$$(r(t)[(x(t)-p_1(t)x(\eta_1(t))+p_2(t)x(\eta_2(t)))^{\Delta}]^{\gamma})^{\Delta}+f(t,x(\tau_1(t)))+g(t,x(\tau_2(t)))=0,$$

Our results not only generalize some existing results in [22], but also can be applied to some oscillation problems that do not covered before. Also, we give some examples to explain our results. The results of this chapter published in:
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Chapter 5 is concerned with the oscillatory behavior of all solutions of the second-order mixed nonlinear neutral dynamic equation with damping on a time scale $\mathbb{T}$

$$(r(t)\phi(z^{\Delta}(t)))^{\Delta}+p(t)\phi(z^{\Delta}(t))+f(t,x(\tau_1(t)))+g(t,x(\tau_2(t)))=0,$$

where $\phi(s)=|s|^{\gamma-1}s$ and $z(t)=x(t)+p_1(t)x(\eta_1(t))+p_2(t)x(\eta_2(t))$. Our results generalize the results of [22] and [30] which are considered as special cases of our results when taking $\alpha=\beta=\gamma, p(t)=p_2(t)=0$ and considering either $g(t,x(\tau_2(t)))=0$ or $f(t,x(\tau_1(t)))=0$. Also, we introduce an illustrated example to explain our results.

Chapter 1

Preliminaries

This chapter is considered as a background for the subject of this thesis. We give a survey of the related material included in previous studies and present some basic concepts for the theory of functional differential equations. Also, we sketch some preliminary results that can be used in this thesis and introduce some of the recent developments in oscillation theory of second order neutral differential equations.

1.1 Initial Value Problems

In this section, we give the definitions of ordinary and functional differential equations.

Definition 1.1.1 *An ordinary differential equation (ODE) is an equation that contains a function of only one independent variable, and some of its derivatives with respect to that variable.*

Definition 1.1.2 *[45] A functional equation (FE) is an equation involves an unknown function for different argument values. The difference between the argument values of the unknown function and t in the FE are called argument deviations. If all argument deviations are constants, then the FE is called a difference equation.*

Example 1.1.1 *The equations $x(3t)+4t^3x(6t) = 4$, and $x(t) = e^t x(t+1)-[x(t-3)]^2$ are examples of FEs.*

Combining Definition 1.1.1 and Definition 1.1.2, we obtain the following definition of functional differential equation (FDE), or equivalently, differential equation with deviating arguments.

Definition 1.1.3 *[45] A functional differential equation is an equation contains an unknown function and some of its derivatives for different argument values. The order of a FDE is the order of the highest derivative of the unknown function. So, a FE is a functional differential equation of order zero.*

Definition 1.1.4 *The ordinary differential equation*

$$x'(t) = f(t, x(t)) \tag{1.1}$$

together with the condition

$$x(t_0) = x_0, \tag{1.2}$$

is called an initial value problem. Eq. (1.2) is called an initial condition, t_0 is an initial point.

It is well known that under certain assumptions on f, the initial value problem Eqs. (1.1) and (1.2) has the unique solution

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))ds \qquad for \qquad t \geq t_0 \tag{1.3}$$

Definition 1.1.5 *The differential equation of the form*

$$x'(t) = f(t, x(t), x(t-\tau)) \qquad with \qquad \tau > 0 \qquad and \qquad t \geq t_0, \tag{1.4}$$

in which the right-hand side depends on the instantaneous position $x(t)$ and the position at τ units back $x(t-\tau)$, is called an ordinary differential equation with delay or a delay differential equation.

Whenever necessary, we consider the integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\tau))ds, \tag{1.5}$$

equivalent to Eq. (1.4). In order to find a solution of Eq. (1.4), we need to have a known function φ on $[t_0 - \tau, t_0]$, instead of the initial condition $x(t_0) = x_0$.

The basic initial value problem for a delay differential equation on the interval $[t_0, T]$, $T < \infty$, is defined by Eq. (1.4) and the initial condition

$$x(t) = \varphi(t) \qquad for\ all \qquad t \in E_{t_0}, \tag{1.6}$$

where t_0 is an initial point and $E_{t_0} = [t_0 - \tau, t_0]$, and the function φ on E_{t_0} is called the initial function. Usually, it is assumed that $\varphi(t_0 + 0) = \varphi(t_0)$. By a one-sided derivative, we mean the derivative at that side of the interval. Under general assumptions, the existence and uniqueness of the solution of the initial value problem Eqs. (1.4) and (1.6) can be established (see for example Ladas[26]). The solution is sometimes denoted by $x(t, \varphi)$. In case of variable delay $\tau = \tau(t) > 0$, the initial set E_{t_0} has the form:

$$E_{t_0} = \{t_0\} \cup \{t - \tau(t) : t - \tau(t) < t_0, t \geq t_0\}.$$

To determine the solution on the interval $[t_0, T]$, we use the initial set

$$E_{t_0 T} = \{t_0\} \cup \{t - \tau(t) : t - \tau(t) < t_0, t_0 \leq t \leq T\}.$$

Example 1.1.2 *[1] for the equation*

$$x'(t) = f(t, x(t), x(t - cos^2 t)),$$

$t_0 = 0$, $E_0 = [-1, 0]$, *the initial function* φ *must be given on* $[-1, 0]$.

The dependence of E_{t_0} on t_0, can be seen in the following example.

Example 1.1.3 *[1] for the equation*

$$x'(t) = a\, x(t/2),$$

we have $\tau(t) = t/2$ *so that*

$$E_0 = \{0\} \ and\ E_1 = [1/2, 1].$$

Now, consider the differential equation of order n with l deviating arguments,
$x^{(m_0)}(t) = f(t, x(t), ..., x^{(m_0-1)}(t), x(t - \tau_1(t)), ..., x^{(m_1-1)}(t - \tau_1(t)), ...,$

$$x(t - \tau_l(t)), ..., x^{(m_l-1)}(t - \tau_l(t))), \tag{1.7}$$

where the deviations $\tau_i(t) > 0$, and $n = \max\{\max_{1 \leq i \leq l}(m_i - 1), m_0\}$. In order to formulate the initial value problem for Eq. (1.7), each deviation $\tau_i(t)$ defines an initial set $E^{(i)}_{t_0}$ as: