



Geometric properties of some types of locally convex spaces

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Pure Mathematics

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Summary

It is well known that a normed space E is uniformly convex (smooth) normed space if and only if its dual E^* is uniformly smooth (convex). We extend these geometric properties to seminormed spaces and then introduce definitions of uniformly convex (smooth) countably seminormed spaces. A new vision of the completion of countably seminormed space was helpful in our task. We get some fundamental links between Lindenstrauss duality formulas. A duality property between uniform convexity and uniform smoothness of countably seminormed space is also given.

Also we give a definition of countably normed space associated with countably seminormed space with compatible seminorms and a definition of Metric projection in a countably seminormed space.

This M. Sc. thesis is organized as follows:

1. Introduction, we show the importance of locally convex spaces and we give a general view of what we have been done in this thesis.
2. In chapter #1, we give a summary of topological spaces [1], a directed system [1], topological vector spaces [2], Hausdorff topological vector Spaces [2] and quotient topological vector spaces [2] almost of the details needed in this thesis.
3. In chapter #2, we study locally convex spaces [2], seminorms [2] and metrizable topological vector spaces [2].
4. In chapter #3, we study completion of different spaces (Metric space [5], normed space by using the technique of associated

normed space of a seminormed space which is the associated Hausdorff space of a seminormed space and countably normed spaces).

5. In chapter #4, we study uniformly convex Banach space, the modulus of convexity, uniformly smooth Banach space [3, 4, 13] almost of the details needed in this thesis and uniformly convex (smooth) countably normed spaces [9, 18].
6. In chapter #5, we give a brief history of Metric projection existences and uniqueness in different spaces (Hilbert space [3], uniformly convex Banach space [14, 15, 16] and countably normed space [9, 18]).
7. In chapter #6, we give new definitions and prove new results **published** in [17]. In fact, we define new types of spaces, the so called *uniformly convex* and *uniformly smooth countably seminormed spaces*. We discuss some geometric properties for these new spaces and we prove one of the most important results which is “a countably seminormed space E is uniformly convex if and only if all dual spaces $\bar{E}_{p_n}^*$ are uniformly smooth”. It is our purpose in this chapter to first extend the notion of uniform convexity and uniform smoothness for Banach spaces to countably seminormed spaces. Moreover, by extending some theorems for uniformly convex uniformly smooth Banach spaces to the case of countably seminormed space, we prove the existence of the metric projection point in uniformly convex countably seminormed space.

Introduction

With a few exceptions the locally convex spaces are the most general spaces in analysis which contain normed spaces, countably normed spaces and countably seminormed spaces. Locally convex spaces can be divided into two classes. First, there are the normed spaces, which belong to classical functional analysis, and whose theory can be considered essentially closed. The second class consists of the so-called nuclear locally convex spaces, which were introduced in 1951 by A. Grothendieck. The two classes have a trivial intersection, since it can be shown that only finite dimensional locally convex spaces are simultaneously normable and nuclear [1].

In this thesis we have generalized some nice geometric properties and definitions from normed spaces into countably seminormed spaces. Thus, as the definitions of *uniformly convex* and *uniformly smooth* which play a very important role in the theory of normed spaces (see chapter 4). Also, we generalize the notion of *a metric projection* (see chapter 5) in countably seminormed spaces which is very important in the theory of fixed point of non self mapping.

Chapter 1

Topological Vector Spaces

We begin with basic definitions and some lemmas which will be needed in the sequel.

1.1 Topological Spaces and A directed system (Net)

Definition 1 ([1])

A topology τ is a family of subsets given on a set M if to each element $x \in M$, there correspond a non-empty system $\mathbb{U}_\tau(x)$ of subsets which satisfies the following conditions:

(U_1) If $U \in \mathbb{U}_\tau(x)$, then $x \in U$.

(U_2) For finitely many sets $U_1, \dots, U_n \in \mathbb{U}_\tau(x)$ there is a set $U \in \mathbb{U}_\tau(x)$ with $U \subset U_1, \dots, U_n$.

(U_3) Each set $U \in \mathbb{U}_\tau(x)$ contains a subset $\acute{U} \in \mathbb{U}_\tau(x)$ such that for each element $y \in \acute{U}$ there is a set $V \in \mathbb{U}_\tau(y)$ with $V \subset U$.

A set with a topology is called **a topological space**.

Definition 2 ([1])

A subset U of a topological space M is called **a neighborhood of the element** x if there is a set $U_0 \in \mathbb{U}_\tau(x)$ with $U_0 \subset U$. $\mathbb{U}_\tau(x)$ is then referred to as **a fundamental system of neighborhoods of the element** x (**local base of** x).

Definition 3 ([1])

A topology τ_1 is **finer** than a topology τ_2 if each τ_2 -neighborhood of an arbitrary element $x \in M$ is also a τ_1 -neighborhood. The topology τ_2 is

then said to be **coarser** than the topology τ_1 . Two topologies on a set M are **equal** if all elements $x \in M$ have the same neighborhoods.

Definition 4 ([1])

A topology can be introduced on each subset $\acute{M}$ of a topological space M by taking the collection of intersections $\acute{U} = \acute{M} \cap U$ with $U \in \mathbb{U}_\tau(x)$ as a fundamental system of neighborhoods for each element $x \in \acute{M}$. This is called **the topology induced** on $\acute{M}$ by M .

Definition 5 ([1])

A **directed system** (x_α) in a topological space M is a set of elements $x_\alpha \in M$ which are uniquely associated with the elements of an index set A . For certain pairs of indices α and β in A , a relation $\alpha \geq \beta$ (read: α is greater than β) is defined which has the following properties:

(G₁) For $\alpha, \beta, \gamma \in A$, $\alpha \geq \beta$ and $\beta \geq \gamma$ always imply $\alpha \geq \gamma$.

(G₂) For finitely many indices $\alpha_1, \dots, \alpha_n \in A$, there is an index $\alpha \in A$ with $\alpha \geq \alpha_i$, $i = 1, \dots, n$.

Directed systems over the set of natural numbers are called **sequences**.

A directed system (x_α) **converges** to an element $x \in M$, if for each neighborhood $U \in \mathbb{U}_\tau(x)$ there is an index $\alpha_0 \in A$ such that

$$x_\alpha \in U \text{ for } \alpha \geq \alpha_0.$$

This is written

$$\lim_{\alpha} x_\alpha = x,$$

and x is designated as **the limit** of the directed system (x_α) .

Definition 6 ([1])

A subset G of a topological space M is **open** if for each element $x \in G$ there is a set $U \in \mathbb{U}_\tau(x)$ with $U \subset G$. The empty set is considered open.

Definition 7 ([1])

The closed hull $\bar{S}$ of an arbitrary subset S of a topological space M consists of all elements $x \in M$ for which every intersection $S \cap U$ with $U \in \mathbb{U}_\tau(x)$ is non-empty. Each subset is contained in its closed hull. Those subsets which coincide with their closed hull are said to be **closed**. The empty set is considered closed as well as open.

Definition 8 A subset D of a topological space M is **dense** in M if its closed hull $\bar{D}$ coincides with M . A topological space is **separable** if it contains a countable dense subset.

1.2 Basic definitions

Let E be a vector space over the field of complex numbers $\mathbb{C}$ (in short, a vector space). Let

$$\mathbb{A}_v : E \times E \rightarrow E, \quad (x, y) \mapsto x + y,$$

$$\mathbb{M}_s : \mathbb{C} \times E \rightarrow E, \quad (\lambda, x) \mapsto \lambda x,$$

be the vector addition and the scalar multiplication in E .

Definition 9 ([2])

A topology $\mathfrak{T}$ on E is said to be **compatible with the linear structure of E** if $\mathbb{A}_v$ and $\mathbb{M}_s$ are continuous when we provide E with the topology $\mathfrak{T}$, $E \times E$ with the product topology $\mathfrak{T} \times \mathfrak{T}$, and $\mathbb{C} \times E$ with the product topology $\mathfrak{L} \times \mathfrak{T}$, where $\mathfrak{L}$ is the usual topology in the complex plane $\mathbb{C}$.

E becomes a **topological vector space** ("TVS"), if it is provided with a topology compatible with its linear structure.

Proposition 10 ([2])

The fundamental system $\mathbb{U}_\tau(x)$ of neighborhoods of the element x is the family of sets $V + x$, where V varies over the fundamental system of neighborhoods of the neutral element, $\mathbb{U}_\tau(0)$.

Definition 11 ([2])

A subset A of a vector space E is said to be **absorbing** if to every $x \in E$ there is a number $c_x > 0$ such that for all $\lambda \in \mathbb{C}$ with $|\lambda| \leq c_x$, we have $\lambda x \in A$.

Definition 12 ([2])

A subset A of a vector space E is said to be **balanced** if for every $x \in A$ and every $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$, we have $\lambda x \in A$.

Theorem 13 ([2])

A fundamental system $\mathbb{U}_\tau(0)$ on a vector space E is compatible with the

linear structure of E if and only if it has the following properties:

- 1-The origin belongs to every subset U belonging to $\mathbb{U}_\tau(0)$.
- 2-To every $U \in \mathbb{U}_\tau(0)$ there is $V \in \mathbb{U}_\tau(0)$ such that $V + V \subset U$.
- 3-For every $U \in \mathbb{U}_\tau(0)$ and for every $\lambda \in \mathbb{C}$ with $\lambda \neq 0$, we have $\lambda U \in \mathbb{U}_\tau(0)$.
- 4-Every $U \in \mathbb{U}_\tau(0)$ is absorbing.
- 5-Every $U \in \mathbb{U}_\tau(0)$ contains some $V \in \mathbb{U}_\tau(0)$ which is balanced.

Any neighborhood of zero is a union of balanced sets which are neighborhood of zero.

If E is "TVS", then there exist a basis of neighborhood of zero of balanced sets.

Proposition 14 ([2])

There is a basis of neighborhoods of zero in a "TVS" E which consists of closed sets.

Proof: It suffices to show that an arbitrary neighborhood U of zero in E contains a closed neighborhood of 0. Let V be another neighborhood of 0 such that $V - V \subset U$. Claim $\bar{V} \subset U$. Indeed, let $x \in \bar{V}$, which means that every neighborhood of x , in particular $V + x$, meets V . Thus, there are elements $y, z \in V$ such that $z = x + y$. In other words,

$$x = z - y \in V - V \subset U.$$

Corollary 15 ([2])

There is a basis of neighborhoods of 0 in E consisting of closed balanced sets.

Proof: Every neighborhood U of 0 in E contains a closed neighborhood of 0, V , which contains a balanced neighborhood W of 0. Then $\bar{W}$ is closed and balanced; $\bar{W} \subset V \subset U$.

Proposition 16 ([2])

In a "TVS" E , if a vector subspace M is open, we have $M = E$.

Every proper subspace of TVS E is closed.

Proof: M being open is a neighborhood of each one of its points, in particular of the origin, hence must be absorbing. But, if $\lambda x \in M$ with $\lambda \neq 0$, then $x = \lambda^{-1}(\lambda x) \in M$.

Definition 17 ([2])

We call a directed system (x_α) from a "TVS" E , a **Cauchy-system**,

if for each zero neighborhood $U \in \mathbb{U}_\tau(0)$ there is an index α_0 with

$$x_\alpha - x_\beta \in U \text{ for } \alpha, \beta \geq \alpha_0.$$

Each convergent directed system is a Cauchy-system. If the converse of this statement is also valid then the "TVS" is called **complete**.

1.3 Hausdorff Topological Vector Spaces

Definition 18 ([2])

A topological space X is said to be **Hausdorff** if given any two distinct points x and y of X , there is a neighborhood U of x and a neighborhood V of y which do not intersect, i.e., such that $U \cap V = \emptyset$.

A very important property of Hausdorff topological spaces is the so-called uniqueness of the limit:

Lemma 19 ([2])

A directed system (x_α) with $\alpha \in A$, on a Hausdorff topological space X , converges to at most one point.

Proof: Suppose that (x_α) with $\alpha \in A$ on X would converge to two distinct points x and y . Let U (resp. V) be a neighborhood of x (resp. y) such that $U \cap V = \emptyset$. Then there exist $\alpha_1 \in A$ such that $x_\alpha \in U$ for $\alpha \geq \alpha_1$ and $\alpha_2 \in A$ such that $x_\alpha \in V$ for $\alpha \geq \alpha_2$. Hence their intersection is nonempty which is a contradiction.

Lemma 20 ([2])

In a Hausdorff space, any set consisting of a single point is closed.

Proof: Using the definition of limit point.

Remark 21 ([2])

There are topological spaces with the same property which are not Hausdorff; but such spaces are not "TVS", as will be seen.

Definition 22 ([2])

Let E be a non empty set. A topology $\tau = \{U \subset E : \text{the complement of } U \text{ is a finite subset of } E\}$ is called the *Cofinite topology* on E .

Example 23 ([2])

Cofinite topology on infinite set.

Proposition 24 ([2])

A "TVS" E is Hausdorff if and only if to every point $x \neq 0$ there is a neighborhood U of 0 such that $x \notin U$.

Proof: The necessity condition is trivial. Suppose it is satisfied. Let x, y be two distinct points of E , which means that $x - y \neq 0$. Then there is a neighborhood U of 0 such that $x - y \notin U$. Choose a balanced neighborhood V of 0, such that $V + V \subset U$. Since V is balanced, we have $-V = V$, hence $V - V \subset U$. Suppose that the intersection

$$(V + x) \cap (V + y) \neq \emptyset$$

Let z be one of its points: $z = x + x', z = y + y'$, with $x', y' \in V$. We have

$$x - y = y' - x' \in V - V \subset U,$$

which contradicts our choice of U . Thus $(V + x) \cap (V + y) = \emptyset$.

Proposition 25 ([2])

In a "TVS" E , the intersection of all neighborhoods of the origin is a vector subspace of E , which is the closure of the set $\{0\}$.

Corollary 26 ([2])

For a "TVS" E to be Hausdorff, it is necessary and sufficient for the set $\{0\}$ to be closed in E , or that the complement of the origin be open in E .

Indeed, to say that $\{0\}$ is closed in E is equivalent to say that $N = \{0\}$ or that no point $x \neq 0$ may belong to all the neighborhoods of 0.

Proposition 27 ([2])

Let f, g be two continuous mappings of a topological space X into a Hausdorff "TVS" E . The set A in which f and g coincide,

$$A = \{x \in X; f(x) = g(x)\},$$

is closed in X .

Proof: A is the preimage of the closed set $\{0\} \subset E$ under the continuous mapping $x \mapsto f(x) - g(x)$.

Proposition 28 ([2])

Let X, E, f, g be as in the previous proposition. If f and g are equal on a dense subset Y of X , they are equal everywhere in X .

Proof: $f = g$ on a closed subset of X (previous proposition) containing Y .

1.4 Quotient Topological Vector Spaces

Let M be a vector subspace of E , and let us consider the quotient vector space E/M and the canonical map $\phi : E \rightarrow E/M$ which assigns to every $x \in E$ its class $\phi(x)$ modulo M or $x + M$.

A topology on E/M which is called the quotient topology on E/M is defined as a subset $\dot{U}$ of E/M is a neighborhood of zero if and only if there is a neighborhood U of zero in E whose image under ϕ is equal to $\dot{U}$, i.e., $\dot{U} = \phi(U)$. The neighborhoods of zero in E/M are the direct images under ϕ of the neighborhoods of 0 in E .

Lemma 29 ([2])

The canonical map ϕ is continuous.

Proof: Let $\dot{U}$ be a neighborhood of the origin in E/M ; there is a neighborhood U of zero in E such that $\phi(U) = \dot{U}$, hence $U \subset \phi^{-1}(\dot{U})$.

Proposition 30 ([2])

Let E be a "TVS", and M a vector subspace of E . The two following properties are equivalent:

- (a) M is closed;
- (b) E/M is Hausdorff.

Corollary 31 ([2])

The "TVS" $E/\overline{\{0\}}$ is Hausdorff.

The "TVS" $E/\overline{\{0\}}$ is called the Hausdorff topological vector space associated with the "TVS" E . When E itself is Hausdorff, $\phi : E \rightarrow E/\overline{\{0\}}$ (canonical mapping) is one-to-one and onto, since then $\overline{\{0\}} = \{0\}$, and $E/\overline{\{0\}}$ is identified with E .

Chapter 2

Locally Convex Spaces and Seminorms.

2.1 Basic definitions

Definition 32 ([2])

A "TVS" E is said to be a **locally convex space** if there is a basis of neighborhoods in E consisting of convex sets.

Locally convex spaces are by far the most important class of "TVS".

Proposition 33 ([2])

In a locally convex space E , there is a basis of neighborhoods of zero consisting of barrels. (A barrel set is convex, balanced and absorbent.)

It suffices to prove that every neighborhood of zero contains a neighborhood of zero which is barrel. Let U_1 be an arbitrary neighborhood of zero in E . Since E is a "TVS", then U_1 contains a closed neighborhood of 0, say V . Since E is locally convex, then V contains a convex neighborhood of zero, W ; and W contains a balanced neighborhood of 0, say U . Since U is balanced, then the set $\bigcup_{|\lambda|} \lambda U = U$. Since W is convex and the convex hull of U is the smallest closed set contains U , then the convex hull of $\bigcup_{|\lambda|} \lambda U = U$ contained in W . Since V is closed, then the closure of the convex hull which is a barrel contained in V , hence in U_1 .