Some Graph Labelings

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CHAPTER ONE

INTRODUCTION

1.1 Some fundamentals in graph theory:

1.1.1 Definitions [21]:

A simple graph G consists of a non-empty finite set $V(G)$ of elements called **vertices** (or nodes), and a finite set $E(G)$ of distinct unordered pairs of distinct elements of $V(G)$ called **edges**. We call $V(G)$ the **vertex set** and $E(G)$ the **edge set** of G. An edge $\{v, w\}$ is said to join the vertices v and w, and is usually abbreviated to vw. For example, Figure 1.1 represents the simple graph G whose vertex set $V(G)$ is $\{u, v, w, z\}$, and whose edge set $E(G)$ consists of the edges uv, uw, vw and wz.

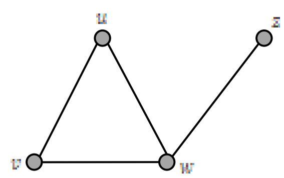


Figure 1.1

In any simple graph there is at most one edge joining a given pair of vertices. However, many results that hold for simple graphs can be extended to more general objects in which two vertices may have several edges joining them. In addition, we may remove the restriction that an edge joins two *distinct* vertices, and allow loops -edges joining a vertex to itself.

The resulting object, in which loops and multiple edges-are allowed, is called a general graph - or, simply, a graph. Thus every simple graph is a graph, but not every graph is a simple graph.

Anyway through chapters 2,3,4 and 5, wherever we say "**graph"** we mean "**simple graph"**.

Also it is essential to mention that we refer to a graph $\boldsymbol{G}$ as $\boldsymbol{G(V(G),E(G))}$ or $\boldsymbol{G(p,q)}$, where

$\boldsymbol{p} = |\boldsymbol{V(G)}|$(= **order of** $\boldsymbol{G}$), $\boldsymbol{q} = |\boldsymbol{E(G)}|$(= **size of** $\boldsymbol{G}$).

Isomorphism

Two general graphs G_1 and G_2 are **isomorphic** if there is a one-one correspondence between the vertices of G_1 and those of G_2 such that the number of edges joining any two vertices of G_1, is equal to the number of edges joining the corresponding vertices of G_2. Thus the two graphs shown in Figure 1.2 are isomorphic under the correspondence $u \leftrightarrow l, v \leftrightarrow m, w \leftrightarrow n, x \leftrightarrow p, y \leftrightarrow q, z \leftrightarrow r$

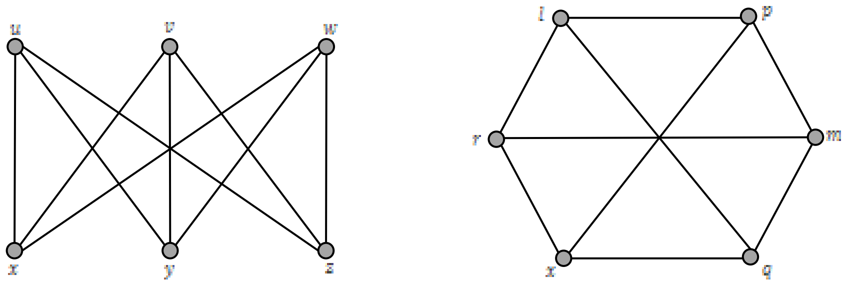


Figure 1.2

Connectedness

We can combine two graphs to make a larger graph. If the two graphs are $G_1 = \left(V(G_1), E(G_1)\right)$ and $G_2 = (V(G_2), E(G_2))$, where $V(G_1)$ and $V(G_2)$ are disjoint, then their **union** $\boldsymbol{G_1 \cup G_2}$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge family $E(G_1) \cup E(G_2)$ (see Figure 1.3).

Thus a graph is **connected** if it cannot be expressed as the union of two graphs, and **disconnected** otherwise. Clearly any **disconnected** graph G can be expressed as the union of connected graphs.

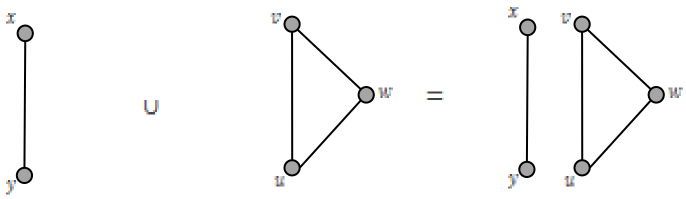
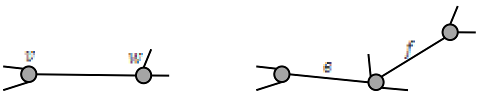


Figure 1.3

Adjacency

We say that two vertices v and w of a graph G are **adjacent** if there is an edge vw joining them, and the vertices v and w are then **incident** with such an edge. Similarly, two distinct edges e and f are **adjacent** if they have a vertex in common (see Figure 1.4).



Adjacent vertices **Adjacent edges**

Figure 1.4

The **degree** of a vertex v of G is the number of edges incident with v, and is written deg(v); in calculating the degree of v, we usually make the convention that a loop at v contributes 2 (rather than 1) to the degree of v. A vertex of degree 0 is an **isolated vertex** and a vertex of degree 1 is an **end-vertex.**

Note: *in any graph the sum of all the vertex-degrees is an even number* - in fact, twice the number of edges, since each edge contributes exactly 2 to the sum. This result is called the **handshaking lemma.** An immediate corollary of the handshaking lemma is that *in any graph the number of vertices of odd degree is even.*

Subgraphs

A subgraph of a graph G is a graph, each of whose vertices belongs to $V(G)$ and each of whose edges belongs to $E(G)$. Thus in Figure 1.5 the graph (a) is a subgraph of the graph (b), but is not a subgraph of the graph (c), since the latter graph contains no 'triangle= C_3'.

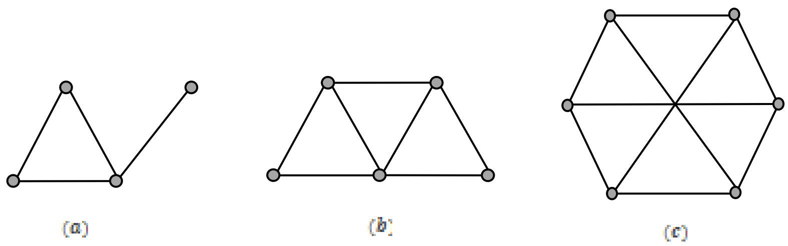


Figure 1.5

1.1.2 Some important types of graphs [21]

Null graphs

A graph whose edge-set is empty is a null graph. We denote the null graph on n vertices ,(*of "**order**"* n), by N_n or $\overline{K}_n$.

Complete graphs

A simple graph in which each pair of distinct vertices are adjacent is a complete graph. We denote the complete graph of order n by K_n. Realize that K_n has $\binom{n}{2} = n(n - l)/2$ edges.

Cycle graphs and path graphs

A connected graph that is regular of degree 2 is a cycle graph. We denote the cycle graph of order n by C_n. The graph obtained from C_n by removing an edge is the path graph of order n, denoted by P_n. The graphs C_6 and P_6 are shown in Figure 1.6. $\boldsymbol{C_3}$ is called **Triangle**.

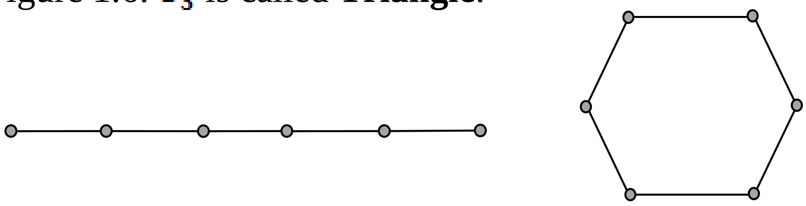


Figure 1.6

Bipartite graphs

If the vertex set of a graph G can be split into two disjoint sets A and B so that each edge of G joins a vertex of A and a vertex of B, then G is a **bipartite graph**. Alternatively, a bipartite graph is one whose vertices can be coloured black and white in such a way that each edge joins a black vertex (in A) and a white vertex (in B). A **complete bipartite graph** is a bipartite graph in which each vertex in A is joined to each vertex in B by just one edge. We denote the bipartite graph with r black vertices and s white vertices by $K_{r,s}$; $K_{1,3}$, $K_{2,3}$ and $K_{3,3}$ are shown in Figure 1.7. Note that $K_{r,s}$ has $r + s$ vertices and rs edges.

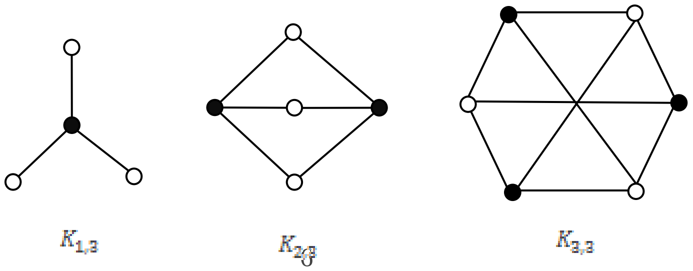


Figure 1.7

It is useful here to mention the lemma stating that every bipartite graph contains no **odd cycles** (= cycles of odd order).

As a similar argument of a bipartite graph, a graph G is ***k*-partite graph**, $k \geq 3$, if it is possible to partition $V(G)$ into k subsets $V_1, V_2, \ldots, V_k$ (called partite sets) such that every element of $E(G)$ joins a vertex of V_i to a vertex of V_j, $i \neq j$.

A **complete *k*-partite graph** G is a k-partite graph with partite sets $V_1, V_2, \ldots, V_k$, having the added property that if $u \in V_i$ and $v \in V_j$, $i \neq j$, then $uv \in E(G)$. If $|V_i| = n_i$, then this graph is denoted by $\boldsymbol{K_{n_1,n_2,\ldots,n_k}}$.(*see* $K_{1,1,m}$ *in Figure 2.15*).

Trees

A **forest** is a graph that contains no cycle and a connected forest is a **tree.**

The complement of a simple graph

If G is a simple graph with vertex set $V(G)$, its **complement** $\overline{\boldsymbol{G}}$ is the simple graph with vertex set $V(G)$ in which two vertices are adjacent if and only if they are *not* adjacent in G.

1.1.3 Defintions of two operations on graphs [7]

1) The **corona** $G_1 \odot G_2$ of two graphs G_1 and G_2 was defined by Frucht and Harary as the graph H obtained by taking one copy of G_1 (which has p vertices) and p copies of G_2, and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.(*See Figure 1.8*).

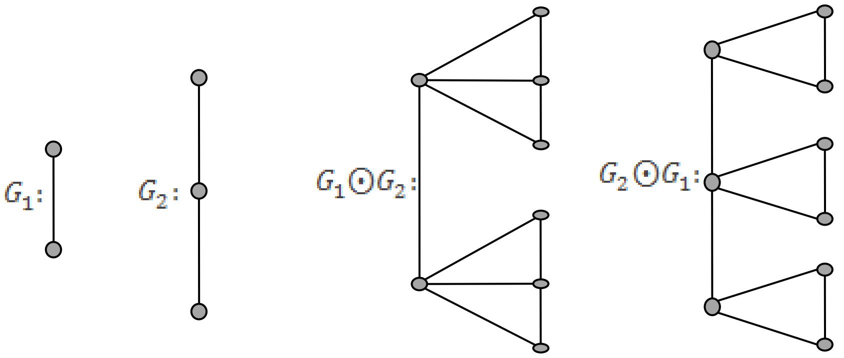


Figure 1.8 Two graphs and their coronas

2) The **cartesian product** $G = G_1 \times G_2$ has $V(G) = V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either

$u_1 = v_1$ and $u_2 v_2 \in E(G_2)$ or

$u_2 = v_2$ and $u_2 v_2 \in E(G_1)$. *See Figure 1.9*

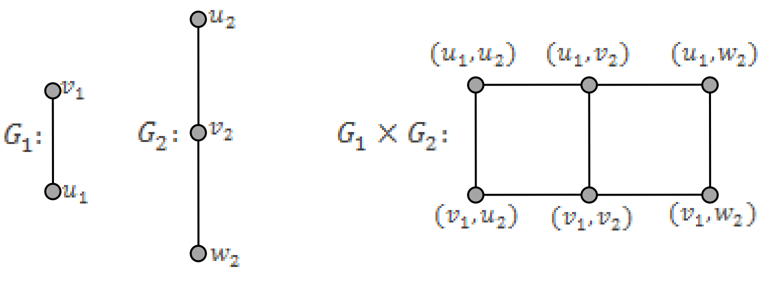


Figure 1.9 The product of two graphs

1.2 Introduction to graph labeling

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions.

Most graph labeling methods trace their origin to one introduced by Rosa [14]in 1967, or one given by Graham and Sloane [6] in 1980. Rosa [14]called a function f a β-valuation of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. Golomb [5] subsequently called such labelings graceful and this is now the popular term. Rosa introduced β-valuations as well as a number of other labelings as tools for decomposing the complete graph into isomorphic subgraphs. In particular, β-valuations originated as a means of attacking the conjecture of Ringel [13] that K_{2n+1} can be decomposed into $2n + 1$ subgraphs that are all isomorphic to a given tree with n edges. Although an unpublished result of Erdös says that most graphs are not graceful (cf. [6]), most graphs that have some sort of regularity of structure are graceful. Sheppard [15] has shown that there are exactly $q!$ gracefully labeled graphs with q edges. Balakrishnan and Sampathkumar [2] have shown that every graph is a subgraph of a graceful graph. Rosa [14]has identified essentially three reasons why a graph fails to be graceful: (1) G has "too many vertices" and "not enough edges," (2) G "has too many edges," and (3) G "has the wrong parity." An infinite class of graphs that are not graceful for the second reason is given in [1]. As an example of the third condition Rosa [14]has shown that if every vertex has even degree and the number of edges is congruent to 1 or 2 (mod 4) then the graph is not graceful. In particular, the cycles C_{4n+1} and C_{4n+2} are not graceful.

Harmonious graphs naturally arose in the study by Graham and Sloane [6] of modular versions of additive bases problems stemming from error-correcting codes. They defined a graph G with q edges to be harmonious if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x) + f(y)$ (mod q), the resulting edge labels are distinct. When G is a tree, exactly one label may be used on two vertices. Analogous to the "parity" necessity condition for graceful graphs, Graham and Sloane proved that if a harmonious graph has an even number q of edges and the degree of every vertex is divisible by 2^k then q is divisible by 2^{k+1}. Liu and Zhang [10]have generalized this condition as follows: if a harmonious graph with q edges has degree sequence $d_1, d_2, \ldots, d_p$ then gcd$(d_1, d_2, \ldots d_p, q)$ divides $q(q-1)/2$. They have also proved that every graph is a subgraph of a harmonious graph. Determining whether a graph has a harmonious labeling was shown to be NP-complete by Auparajita, Dulawat, and Rathore in 2001 (see [9]).

Over the past three decades in excess of 800 papers have spawned a bewildering array of graph labeling methods. Despite the unabated procession of papers, there are few general results on graph labelings. Indeed, the papers focus on particular classes of graphs and methods, and feature ad hoc arguments.

Through chapters 2,3,4 and 5 we presents new results in three types of graph labelings as follows: chapter two: new results in mean labeling, which are accepted for publication in the international Canadian journal Ars Combinatoria. Chapter three: new results in super mean labeling, chapter four: new results in prime cordial labeling and chapter five: new results in permutation labeling. These three groups of results are submitted for publication.

We introduce each of these chapters by the definition of the corresponding labeling, and the results which are achieved before, and our new results to be illustrated within the chapter.

CHAPTER TWO

ON MEAN GRAPHS

2.1 Introduction:

Somasundaram and Ponraj introduced the notion of mean labelings of graphs, and they gave results concerning it in several papers [11,16,17,18,19]. A graph $G = (V(G), E(G))$ is said to be a mean graph, if there is an injective function f from $V(G)$ to $\{0, 1, 2, \ldots, |E(G)|\}$, such that when each edge uv is labeled with $\lceil (f(u) + f(v))/2 \rceil$, where $\lceil x \rceil$ denotes the smallest integer not less than x, then the resulting edge labels are distinct.

2.1.1 Example:

In *Figure 2.1* we present the graph C_4 labeled as a mean graph. Realize that we use the labels $\{0,2,3,4\}$, from the set $\{0,1,2,3,4 = |E(G)|\}$, to label the vertices, so the edge labels are distinct (*they are exactly* $\{1,2,3,4\}$).

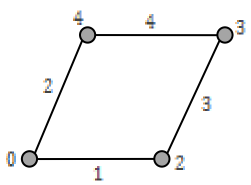


Figure 2.1