Ain Shams University
Faculty of Science
Mathematics Department

On Smooth Bitopological Spaces

Thesis

Submitted to Department of Mathematics, Faculty of Science, Ain Shams University for the award of the Degree of Doctor of Philosophy in Science
(Pure Mathematics)

By

Rasha Naser Majeed

M. Sc. Baghdad University, Iraq 2000
B. Sc. Baghdad University, Iraq 1997
Department of Mathematics, Faculty of Education for Pure Science (Ibn Al-Haitham), Baghdad University, Baghdad-Iraq

Supervisors

Prof. Dr. Osama A. EL-Tantawy
Professor of Pure Mathematics, Faculty of Science, Zagazig University

Prof. Dr. Sobhy A. EL-Sheikh
Professor of Pure Mathematics, Faculty of Education, Ain Shams University

Dr. Fayza A. Ibrahim
Assistant Professor of Pure Mathematics, Faculty of Science, Ain Shams University

(2015)

Ain Shams University

Faculty of Science
Mathematics Department

Candidate: Rasha Naser Majeed

Thesis Title: On Smooth Bitopological Spaces

Degree: Ph.D Degree in Science

(Pure Mathematics)

Supervisors:

No.	Name	Profession	Signature
1.	**Prof. Dr. O. A. El-Tantawy**	Professor of Pure Mathematics, Mathematics Department, Faculty of Science, Zagazig University.	
3.	**Prof. Dr. S. A. El-Sheikh**	Professor of Pure Mathematics, Mathematics Department, Faculty of Education, Ain Shams University.	
2.	**Dr. F. A. Ibrahim**	Assistant Professor of Pure Mathematics, Mathematics Department, Faculty of Science, Ain Shams University.	

Acknowledgements

First of all gratitude and thanks to gracious **Allah** who always helps and guides me. I would like to thank **the prophet Mohamed** "peace be upon him" who urges us to seek knowledge and who is the teacher of mankind. I would like also to thank the supervision committee who are:

Prof. Dr. Osama Tantawy, Professor of Pure Mathematics, Faculty of Science, Zagazig University, who provided me with guidance and continuous encouragement. He learned me many things not only on the scientific side but also in practical and personal life. He did his best for the success of this work through seminars which were held, many discussions, precious comments, valuable reviews and remarks. His efforts during revision of this thesis is an invaluable.

Prof. Dr. Sobhy Ahmed Aly El Sheikh, Professor of Pure Mathematics, Faculty of Education, Ain Shams University, who helped me at the first step in this study through his suggestions for the research problems, valuable instructions, guidance and continuous follow up in this study. He offered me much of his precious time and provided me with his wisdom and knowledge through many discussions we had. Thank you very much my supervisors for your unconditional help and it is really very much appreciated.

Dr. Fayza A. Ibrahim, Assistant Professor of Pure Mathematics, Faculty of Science, Ain Shams University. She helped me humbly with no expectation of return and was always available whenever I wanted a discussion or needed her advice and cooperation during preparing the thesis.

I would like to express my deepest sense of **Prof. Dr. Abd El-Aziz E. Radwan**, the former head of the Mathematics Department, Faculty of Science, Ain Shams University, who offered his continuous helpful support and encouragement whenever I was in need.

I would like to express my deepest sense of **Prof. Dr. Salah Eldien S. Hussein**, the current head of Mathematics Department, Faculty of Science, Ain Shams University.

I extend also my heartfelt thanks to scientific research fellows at the topological school headed by **Prof. Dr. Ali Kandil Saad**.

I am grateful to the Ministry of Higher Education and Scientific Research in Iraq and Egypt for granting me a scholarship for four years. It is pleasure to express my great thanks to the staff of Department of Mathematics, Faculty of Science, Ain Shams University and all colleagues of topological seminar, for their interest, constant encouragement and active discussion.

I want to express my gratitude and deepest appreciation to my sister, **Dr. Saba Naser Majeed** for her understanding, encouragement and personal attention during my Ph.D study.

I am thankful to my colleagues at the Department of Mathematics, Faculty of Education for Pure Science (Ibn Al-Haitham), Baghdad University, Iraq.

I am appreciative to my family, specially my mother, my brothers and my sisters. They were always supporting me and encouraging me with their best wishes. I offer my sincere gratitude to my husband **Sadeq,** for his devotion, sacrifice and continued support me. To my dear children **Baneen, Jaafer, Sara** and **Hawraa,** thank you for supporting through the good times and bad.

Rasha Naser

Contents

Summary

Fuzzy sets are a generalization of conventional set theory that were introduced by Zadeh [91] in 1965 as a mathematical way to represent uncertainty and vagueness and to provide formalized tools for dealing with the imprecision intrinsic to many problems in everyday life. The notion of a fuzzy set has caused great interest among both 'pure' and applied mathematicians. It has also raised enthusiasm among some engineers, biologists, psychologists, economists, and experts in other areas, who use (or at least try to use) mathematical ideas and methods in their research.

After the discovery of the fuzzy sets, much attention has been paid to generalize the basic concepts of classical topology in fuzzy setting and thus a modern theory of fuzzy topology is developed. Chang [13] in 1968 introduced the notion of fuzzy topology when he replaced sets by fuzzy sets in the ordinary definition of topology. Since then, a lot of contributions to the development of fuzzy topology have been published (cf.[24, 25, 28, 58, 59, 60, 64]). In 1980, Pu and Liu [65, 66] introduced the concept of quasi-coincidence relation which done further research on Chang's concept of fuzzy topology.

However, Chang definition for fuzzy topology was criticized by some authors, that his notion did not really describe fuzziness with respect to the concept of openness of a fuzzy set, which seems to be a drawback in the process of fuzzification of the concept of topological spaces. In the light of this difficulty, many mathematicians try to make a fuzzy treatment for this structures. Independently by Kubiak [47] and Šostak [84] in 1985 introduced the fundamental concept of a 'fuzzy topological structure', as an extension of both crisp topology and Chang's fuzzy topology, according to which a fuzzy topology on a set X is a mapping on the power set I^X (i.e., a mapping $\tau : I^X \longrightarrow I$ where $I = [0,1]$

is the closed unite interval) satisfying certain axioms. With respect to this mapping each fuzzy set $\mu \in I^X$ is open with a suitable degree $\tau(\mu)$, i.e., each fuzzy subset has a degree of openness, in the sense that not only the object were fuzzified, but also the axiomatics. Šostak gave some rules and showed how such an extension can be realized. Every fuzzy topology in Šostak sense is a fuzzy topology in Chang sense if $\tau : I^X \longrightarrow \{0, 1\}$. Subsequently, Badard [8] in 1986, introduced the concept of 'smooth topological space'. In 1992, Chattopadhyay et al. [14] and Chattopadhyay and Samanta [15] in 1993, re-introduced the same concept, calling it 'gradation of openess'. In 1992, Ramadan [70] and his colleagues introduced a similar definition, namely, smooth topological space for lattice $L = [0, 1]$. Following Ramadan, several authors have re-introduced and further studied smooth topological space (cf.[6, 21, 43, 71, 72, 80, 85]). Thus, the terms 'fuzzy topology', in Šostak's sense, 'gradation of openness' and 'smooth topology' are essentially referring to the same concept. In our thesis, we adopt the term smooth topology.

The concept of bitopological spaces (X, τ_1, τ_2) was first introduced by Kelly [37] in 1963, as a method of generalizing topological spaces (X, τ). In 1989, Kandil [33] introduced and studied the notion of fuzzy bitopological spaces as a natural generalization of fuzzy topological spaces. In 2001, Lee et al. [53] introduced the concept of smooth bitopological spaces as a generalization of smooth topological spaces and Kandil's fuzzy bitopological spaces. Thereafter, a large number of papers have been written to generalize fuzzy topological concepts to smooth bitopological setting. Ghanim et al. [30] in 2000, introduced the notion of supra smooth topology. Abbas [1] in 2002, generated the supra smooth topological space (X, τ_{12}) from a smooth bitopological space, as an extension of generated supra fuzzy topology in the sense of Kandil et al. [35]. This induced supra smooth topological space plays an important role in the researching of the study of smooth bitopological spaces, because in this case, smooth bitopological spaces studied by one supra smooth topology τ_{12} such that $\tau_i \leq \tau_{12}, i = 1, 2$ and this is the easiest.

The general notion of fuzzy filter on a set X was introduced by Gähler in [17] in 1992, as a mapping of the set L^X of all fuzzy sets in X into the completely distributive complete lattice L. By means of the notion of interior of a fuzzy set, the notion of a fuzzy neighborhood filter is defined. In this approach several notions are related to usual points, e.g. the notion of fuzzy neighborhood filter. Several concepts in fuzzy topological spaces are studied by the notion of fuzzy neighborhood filters (cf. [10, 11, 24, 25, 26, 27]).

Gradation of proximity on a set X is a mapping $\delta : I^X \times I^X \longrightarrow I$ satisfying certain axioms, was introduced by Samanta [81] in 1995. Samanta has showed that his fuzzy proximity is more general than that of Artico and Moresco [5]. Also, in the same year Ghanim et al. [29] introduced gradation of proximity spaces with somewhat different definition of Samanta [81]. Later, Ghanim et al. [30] introduced the concept of gradation of supra proximity. Kim and Park [40] introduced the concept of fuzzy quasi-proximity spaces in view of Ghanim et al. [29].

This thesis which contains six chapters is devoted to give further study for smooth bitopological spaces. The outlines of this study as follow:

(1) Study smooth bitopological spaces by using the induced supra smooth topological spaces.

(2) Introduce and study some types of generalized fuzzy closed sets.

(3) Introduce new types of generalized fuzzy continuous mappings on smooth bitopological spaces.

(4) Generalize separation axioms [10, 11] from fuzzy topological spaces to fuzzy bitopological spaces and then to smooth topological spaces.

(5) Introduce new types of compactness and compactness modulo smooth ideals in smooth bitopological spaces.

(6) Development the theory of fuzzy proximity more than one direction by introducing the notion of smooth biproximity spaces and some other new types of smooth proximity spaces.

Chapter 1, is divided into two sections. Section 1.1, is the introductory section, includes some preliminary results from fuzzy sets, fuzzy filters, fuzzy topological (resp. bitopological, smooth topological, smooth bitopological and fuzzy proximity) spaces. In Section 1.2, we extend the notion of r-fuzzy semi-open sets on smooth bitopological space by using its associate supra smooth topological space (X, τ_{12}). Some of its basic properties are discussed. We saw that the present notion of r-fuzzy semi-open sets is independent from the notion of r-(τ_i, τ_j)-fuzzy semi-open sets in [73] and from r-fuzzy semi-open sets in $(X, \tau_i), i = 1, 2$ [45, 52], several examples are given to clarify these results. In addition, this new class of r-fuzzy semi-open sets construct a new type of supra fuzzy closure operator and then create a new supra smooth topology finer than τ_{12} (see Theorem 1.2.5). Finally, different types of fuzzy semi-continuity are introduced and studied.

Chapter 2, is devoted to introduce and study the notion of r- generalized (resp. r-θ-generalized) fuzzy closed sets in smooth bitopological spaces. Each notion is presented by two ways. By means in the first way the notion introduced in (X, τ_1, τ_2) by using $\tau_i, \tau_j, i, j = 1, 2, i \neq j$. In the second way the induced supra smooth topological space (X, τ_{12}) is used to introduced the notion for smooth bitopological space. The notion of r-generalized (resp. r-θ-generalized) fuzzy closed sets that defined in smooth bitopological space by the first way is independent from that defined by the second way (see Examples 2.2.1, 2.2.2, 2.5.5 and 2.5.6). This means we got four new classes of r-generalized fuzzy closed sets in smooth bitopological space. The relationships between the present notions of r-generalized fuzzy closed sets, r-θ-generalized fuzzy closed sets and the notions that introduced in [43, 44] are given with many examples. Furthermore, each class of r-generalized (resp. r-θ-generalized) fuzzy closed sets that defined by the first way is used to define a new generalized fuzzy closure operator on X and a new smooth topology on X. While class of r-generalized (resp. r-θ-generalized) fuzzy closed sets that defined by the second way is used to define a new generalized supra fuzzy closure operator on X and a new supra smooth topology on X.

Chapter 3, is an application part of the concepts presented in Chapter 2. In this chapter, we introduce and study the concept of (i, j)-GF-continuous (resp. irresolute) mappings. Also, a new class of fuzzy mappings, namely (i, j) strongly-θ-fuzzy continuous (resp. (i, j)-θ-GF-continuous and (i, j)-θ-GF-irresolute) mappings are introduced, basic properties and the relations between them are studied. Furthermore, some types of FP^*-θ-continuous mappings, namely almost strongly θ-continuous (resp. θ-continuous and weakly θ-continuous) mappings are introduced in the light of the concepts of quasi-coincidence and the notion of C^{θ}_{12}-fuzzy closure operator, and some of their properties have investigated. Also, we introduce GFP^*-continuous (resp. open, closed and irresolute) mappings, GFP^*-θ-continuous (resp. irresolute) mappings and FP^*-strongly-θ-continuous mappings on smooth bitopological spaces and discuss their properties in detail. Finally, all the possible relationships between all types mentioned above have been studied with many examples of reverse.

Chapter 4, the first aim of this chapter is introduced the concept of fuzzy pairwise separation axioms $FPT_i, i = 0, 1, 2, 3, 4$ in fuzzy bitopological spaces as a generalized to separation axioms [10, 11] in Chang's fuzzy topological spaces. These separation axioms are related to usual points, ordinary subsets and also a fuzzy neighborhood filter of G*ä*hler [17]. These presented separation axioms are good extension in the sense of Lowen [59]. It is important to note that in [10, 11], the authors pointed out some incorrect results and then used it to show the implication between these axioms, i.e., every T_i-space is T_{i-1}-space $(i = 3, 4)$, in Remark 4.1.2 we see that this results is not true in general. In our study, the condition for which T_i-space $\Longrightarrow T_{i-1}$ $(i = 3, 4)$ is given and we pointed out these incorrect results in [87].

The second aim of this chapter is to extend the separation axioms in [10, 11] to smooth topological (resp. bitopological) spaces which is the focus of our thesis. In order to achieve this, we extend the notion of fuzzy neighborhood filter of G*ä*hler [17] to smooth topological spaces (see Definition 4.2.1), and used it to define separation axioms $T_i, i = 0, 1, 2, 3, 4$. These separation axioms are good extension in sense of Ayg*ü*n et al. [6]. Moreover, the initial smooth topological space of a family of T_i-spaces, $i = 0, 1, 2, 3, 4$, is also a T_i-space. Finally, the relationship between T_i-axioms $i = 0, 1, 2, 3, 4$ are obtained with some examples for clarification.

Chapter 5, will be dedicated to the study of the concept of compactness in smooth bitopological spaces. We use the concept of r-(τ_i, τ_j)-fuzzy semi-open sets and smooth ideals to introduce new types of compactness, namely $FPS\mathcal{I}$-compact, $FPS\mathcal{I}$-Lindelöf, $FP\mathcal{I}$-S-closed and FP-countably $S\mathcal{I}$-compact that generalize to FPS-compact, FPS-Lindelöf, FP-S-closed and FP-countably S-compact respectively. As will as we have $FPSC(\mathcal{I})$-compact which is not a generalize for $FPSC$-compact (see Example 5.2.1). The relationship between these types of compactness and those introduced in [78] are studied. Also, the properties and characterizations are investigated. Finally, the behavior of these types of compactness under some types of mappings is also investigated.

Chapter 6, concern to the concept of fuzzy proximity theory. We introduce the notion of smooth biproximity space (X, δ_1, δ_2), where $\delta_i, i = 1, 2$ is a gradation proximity in the sense of Ghanim et al. [29]. Then, we generate a supra smooth proximity space (X, δ_{12}) from (X, δ_1, δ_2). We discuss the supra smooth topological structure $(X, \tau_{\delta_{12}})$ based on this supra smooth proximity. It will be shown that the induced supra smooth topology $(X, \tau_{\delta_{12}})$ are FR_i-space, $i = 0, 1, 2, 3$ and FT_i-space, $i = 0, 1, 2, 3, 4$ in [75]. Moreover, for each smooth bitopological space which is FP^*T_4, the induced supra smooth topological space is a supra smooth proximal. Moreover, the notion of FP-(resp. FP^*-) proximity map are introduced. In addition, the concept of P-smooth quasi-proximity is introduced and study its basic properties. We discuss the structure of smooth bitopological space based on P-smooth quasi-proximity. Finally, some generalizations of the concept of gradation proximity in view of Ghanim et al. [29] are introduced, precisely a smooth K-proximity space, a Leader and a Lodato smooth proximity spaces. Also, we study some properties of these types.

The results obtained in this Thesis have been published (or accepted for published) in scientific journals as follow:

(1) O. A. Tantawy, S. A. El-Sheikh and R. N. Majeed, r-τ_{12}-fuzzy semiopen sets and fuzzy semi-continuous mappings in smooth bitopological spaces, J. of new Results in Science 6 (2014) 76-94.

(2) O. A. Tantawy, F. Abelhalim, S. A. El-Sheikh and R. N. Majeed, Two new approaches to generalized supra fuzzy closure operators on smooth bitopological spaces, Wulfenia Journal 21 (5) (2014) 221-241.

(3) O. A. Tantawy, S. A. El-Sheikh and R. N. Majeed, r-(τ_i, τ_j)-θ-generalized fuzzy closed sets in smooth bitopological spaces, Gen. Math. Notes 24(1) (2014) 58-73.

(4) Accepted for oral presentation in the 27^{th} Conference of Topology and its Applications (21-22 June 2014), Faculty of Science, Ain Shams University, Egypt.

(5) O. A. Tantawy, F. A. Ibrahim, S. A. El-Sheikh and Rasha N. Majeed, r-(τ_i, τ_j)-generalized fuzzy closed sets in smooth bitopological spaces, Ann. Fuzzy Math. Inform. 9(4) (2015) 537-551.

(6) O. A. Tantawy, R. N. Majeed and S. A. El-Sheikh, r-τ_{12}-θ-generalized fuzzy closed sets in smooth bitopological spaces, J. of New Theory 4 (2015) 60-73.

(7) O. A. Tantawy, S. A. El-Sheikh and R. N. Majeed, C^{θ}_{12}-fuzzy closure on smooth bitopological spaces, J. Fuzzy Math. 23(3) (2015) 559-580.

(8) O. A. Tantawy, S. A. El-Sheikh and R. N. Majeed, Generalized fuzzy closed sets in smooth bitopological spaces, Generalized and Hybrid Set Structures and Applications for Soft Computing, accepted.

(9) O. Tantawy, S. A. El-Sheikh and R. Naser, Fuzzy pairwise separation axioms in fuzzy bitopological spaces, Jökull Journal 63(12) (2013) 243-260.

(10) O. A. Tantawy, S. A. El-sheikh and R. N. Majeed, A Note on "Separation axioms in fuzzy bitopological spaces", J. Intell. Fuzzy Systems 28 (2015) 2243-2244.

(11) Accepted for oral presentation in the 3^{rd} international conference, On Mathematics & information sciences hold in Luxor, Dec. (2013).

(12) O. A. Tantawy, S. A. El-sheikh and R. N. Majeed, T_3 and T_4-spaces in smooth topological spaces, International Journal of Mathematics Trends and Technology 13(1) (2014) 68-78.

(13) O. A. Tantawy, S. A. El-Sheikh and R. N. Majeed, Types of fuzzy pairwise S-compactness modulo smooth ideals, J. of New Theory 3 (2015) 41-66.

(14) O. A. Tantawy, S. A. El-Sheikh and R. N. Majeed, Generalized smooth proximity spaces, Ann. Fuzzy Math. Inform., to appear.

(15) Accepted for oral presentation in the 28^{th} Conference of the Topology and its Applications hold in Mansoura, 28 - 29 July 2015.