AIN SHAMS UNIVERSITY
FACULTY OF WOMEN FOR
ARTS, SCIENCES AND EDUCATION
MATHEMATICS DEPARTMENT

Solving Nonlinear Partial Differential Equations Using Homotopy Analysis Method (HAM)

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BY

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I dedicate this thesis to

My son "Hussein"

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Arabic Summary

Abstract

Abstract

Shaheed Naser Huseen. Solving Nonlinear Partial Differential Equations Using Homotopy Analysis Method (HAM). Doctor of Philosophy in Science dissertation of pure Mathematics, Faculty of Women for Arts, Sciences and Education, Ain Shams University.

The main purpose of this thesis is to find approximate solutions for partial differential equations.

The thesis is divided into five chapters:

In Chapter One, We give the Basic Ideas and Brief History of the Homotopy Analysis Method (HAM) and its optimal methods. We explained the difference between (HAM) and perturbation techniques and non-perturbation methods. So, we give a summary of the some Approaches based on the HAM.

In Chapter Two, We introduced a more general method of homotopy analysis method (HAM) to solve non-linear partial differential equations; it is called q-homotopy analysis method (q-HAM).

The result of this chapter is published in "International Journal of Applied Mathematics and Mechanics" 8 (15), 2012, 51-75.

In Chapter Three, The convergence of q-homotopy analysis method (q-HAM) is studied. It is proved that under certain conditions the solution of the zeroth-order deformation equation

$$(1 - nq)[L(\emptyset(t;q)) - L(u_0)] - qhN[\emptyset(t;q)] = 0,$$

associated with the original problem exists as a power series in q. So, under a special constraint the q-homotopy analysis method does converge to the exact solution of nonlinear problems.

The result of this chapter is published in "International Journal of Contemporary Mathematical Sciences", Vol. 8, no. 10, 2013, 481 – 497.

In Chapter Four, A modified q-homotopy analysis method (mq-HAM) is proposed for solving non-linear partial differential equations. This method improves the convergence of the series solution and overcomes the computing difficulty encountered in the q-homotopy analysis method (q-HAM)

Part of the results of this chapter is published in:

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In Chapter Five: , An optimal q-homotopy analysis method (Oq-HAM) is proposed. This optimal method contains only one convergence-control parameter. The results