On q-Hankel Transforms On Complex Domains

By

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To my late mother

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Notations

$\mathbb{N} := \{1, 2, 3 \cdots\}$

$\mathbb{N}_0 := \{0, 1, 2, \cdots\}$

$\mathbb{Z}$ is the set of integers.

$\mathbb{R} := (-\infty, \infty)$

$\mathbb{R}^+ := (0, \infty)$

$\mathbb{C}$ is the set of all complex numbers.

f.s. is an abbreviation for fundamental set.

Preface

This thesis is a study of zeros and applications of q-Hankel transforms. These transforms are the integral transforms whose kernels are q-Bessel functions of second and third type. In this setting the integral is the Jackson q-integral, cf. [25]. The q-Bessel functions are first defined by Jackson in [24], see also [15]. However in some literature, the third type q-Bessel function is called Hahn-Exton q-Bessel function, see [37]. The study of zeros of q-Bessel functions of second and third type could be split into two phases. The first is the existence phase and the second is the asymptotics one. As for the existence of zeros of the second type q-Bessel function, this is established by Ismail in [20] who also proves that the zeros are real, simple and interlacing. The same is done by Koelink and Swarttouw in [28] for the third type. The asymptotics of zeros of q-Bessel function of the second type is first conjectured by Ismail in the mentioned article. This has been recently modified and proved by Hayman in [17]. The investigations of the asymptotics of zeros of the q-Bessel function of the third type started with the work of Bustoz and Cardoso [14] for a particular case, namely the q-sine functions. Then, Abreu, Bustoz and Cardoso derived the general asymptotics of these zeros under certain restrictions on q, cf. [4]. The study of zeros of q-Hankel and Fourier transforms defined on complex domains is first considered by Annaby and Mansour, [9], followed by [10] and [11]. These studies focus on conditions that guarantee that the zeros of the considered q-Hankel transforms are only real zeros and to derive their asymptotics. In general, q-Hankel transforms have infinitely many real zeros and a finite number of non real zeros. It is worth mentioning that q-Hankel and Fourier transforms are first considered by Koornwinder and Swarttouw, [29]. The classical sampling theory of signal analysis is concerned with recovering (entire) functions from their values at a discrete set of points. In the theory of communications, this means that analog signals can be treated as digital ones. There are few papers that deal with q-analogs of the existing sampling theorems. This works involve the papers of Abreu, [1, 2]; Ismail and Zayed, [22]; Annaby, [7] and Annaby, Bustoz and Ismail, [8]. The present thesis is devoted to investigate two problems. The zeros of q-Bessel functions and associated q-Hankel transforms. It contains four chapters. The first two chapters are introductory chapters and the last two chapters contain our new results. In the first chapter we give a brief account about q-calculus and notations, the definition of q-Bessel functions, and a brief account about known results of their zeros as well as those of q-Hankel transforms are given in the rest of the chapter. Chapter two is also an outline about the classical sampling theory and some known q-analogs. It started with the classical result of Whittaker (1915), Kotel'nikov (1933) and Shannon (1949), which is known as the Whittaker-Kotelnikov (WKS) sampling theorem. Some q-analogs of this theorem are presented in this chapter as well as an analog of a sampling theorem associated with q-Sturm-Liouville problems. The first new results of this thesis exist in Chapter three. We studied the zeros of q-Hankel transforms using Rouche's and Hurwitz theorems. We applied our results to obtain a q-analog of a theorem of Pólya. We also give comparisons between different approaches used to investigate asymptotics of zeros. We give numerical examples and open problems at the end of the chapter. In Chapter 4 we prove the completeness of systems of the second Jackson q-Bessel function.

Chapter 1

q-Calculus And q-Special Functions

In this chapter we include some q-notations. Also we introduce the q-Bessel functions and then end the chapter with some important known results on basic functions.

1.1 q-Calculus

In this section, we introduce the q-calculus that will be needed throughout the thesis. From now on unless otherwise stated q is a positive number less than 1.

Definition 1.1.1. A subset A of $\mathbb{R}$ is called q-geometric if, $x \in A$ implies that $qx \in A$.

The name geometric comes from the fact that if $A \subseteq \mathbb{R}$ is a q-geometric, then it contains all geometric sequences $\{xq^n : x \in A\}_{n=0}^{\infty}$. Obviously zero is an accumulation point of any q-geometric set and any interval containing zero is q-geometric.

The q-difference operator, which was known to Heine, and may be to Gauss, see [15], is defined by the following.

Definition 1.1.2. Let $f(x)$ be a function, real or complex valued, defined on a q-geometric set $A \subseteq \mathbb{R}$. The q-difference operator is defined by Jackson, [25], via the formula

$$D_q f(x) = \frac{f(x) - f(qx)}{x - qx}, \quad x \in A/\{0\}. \tag{1.1.1}$$

If $0 \in A$, we say that $f(x)$ has q-derivative at zero if the limit

$$\lim_{n\to\infty} \frac{f(xq^n) - f(0)}{xq^n}, \ x \in A \tag{1.1.2}$$

exists and does not depend on x. In this case, we shall denote this limit by $D_q f(0)$.

We would like to mention that in many literatures, the q-difference operator at the point zero is defined to be $f'(0)$, see for example, [37]. We notice that if $0 \notin A$, then the q-difference operator always exists. Moreover, $\lim_{q\to 1} D_q f(x) = f'(x)$, provided that A contains a neighborhood of the point x and f is differentiable at x.

The product rule for the q-derivative is

$$(D_q fg)(x) = f(x)(D_q g)(x) + g(qx)(D_q f)(x). \tag{1.1.3}$$

Jackson in 1910 [26] had introduced the following q-integral denoted by

$$\int_a^b f(x)\, d_q x$$

Definition 1.1.3. Let a, b be any two points defined on a q-geometric set A. The q-integration of f over $[a, b]$ is defined by

$$\int_a^b f(t)\, d_q t := \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t, \tag{1.1.4}$$

where

$$\int_0^x f(t)\, d_q t := \sum_{n=0}^{\infty} (xq^n - xq^{n+1}) f(xq^n), \quad x \in [a, b], \tag{1.1.5}$$

provided that the series at the right-hand side of (1.1.5) converges at $x = a$ and b.

Now, we define some spaces that we work with later on .

Let $L_q^1(0, 1)$ be the space of all complex-valued functions f defined on $(0, 1]$ that satisfy

$$\int_0^1 |f(t)|\, d_q t < \infty.$$

The space $L^1_q(0,1)$ associated with the norm-function definition

$$\|f\| := \int_0^1 |f(t)|\, d_qt,$$

is a Banach space.

By $L^2_q(0,1)$ we mean the space of all complex-valued functions f defined on $(0,1]$ such that

$$\int_0^1 |f(t)|^2\, d_qt < \infty. \tag{1.1.6}$$

The space $L^2_q(0,1)$ associated with the inner product defined by

$$\langle f,g\rangle = \int_0^1 f(x)\ \overline{g(x)}\ d_qx,$$

is a Hilbert space.

The q-shifted factorial, see [15], for $a \in \mathbb{C}$, is defined by

$$(a;q)_n := \begin{cases} 1, & n=0, \\ \prod_{i=0}^{n-1}(1-aq^i), & n=1,2,\dots . \end{cases} \tag{1.1.7}$$

The limit of $(a;q)_n$ as n tends to infinity exists and will be denoted by $(a;q)_\infty$. Moreover, $(a;q)_\infty$ has the following series representation, cf., e.g. [15, p. 11],

$$(a;q)_\infty = \sum_{n=0}^{\infty}(-1)^n q^{\frac{n(n-1)}{2}} \frac{a^n}{(q;q)_n}. \tag{1.1.8}$$

The multiple q-shifted factorial for complex numbers $a_1,\dots,a_k$ is defined by

$$(a_1,a_2,\dots,a_k;q)_n := \prod_{j=1}^{k}(a_j;q)_n. \tag{1.1.9}$$

Let ${}_r\phi_s$ denote the $q-$Hypergeometric series

$${}_r\phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}\ \Big|\ q,z\right) := \sum_{n=0}^{\infty} \frac{(a_1,\dots,a_r;q)_n}{(q,b_1,\dots,b_s;q)_n} z^n (-q^{(n-1)/2})^{n(s+1-r)}. \tag{1.1.10}$$

The θ-function is defined for $z \in \mathbb{C}\setminus\{0\}$, $0 < |q| < 1$ to be

$$\theta(z;q) := \sum_{n=-\infty}^{\infty} q^{n^2} z^n. \tag{1.1.11}$$

The following identity is introduced by C.G.J. Jacobi in 1829, and it is called Jacobi triple product identity, see [15]

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = \left(q^2, -qz, -qz^{-1}; q^2\right)_\infty, \quad z \in \mathbb{C}\setminus\{0\}, \quad 0 < |q| < 1. \tag{1.1.12}$$

Therefore $\theta(z;q)$ has only real and simple zeros at the points $\{-q^{-2n-1},\ n \in \mathbb{Z}\}$ (cf. [11]).

Definition 1.1.4. The q- Gamma function, $\Gamma_q(x)$, and the q-Beta function, $\beta_q(x,y)$, are defined by

$$\Gamma_q(x) := \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \qquad \beta_q(x,y) := \frac{\Gamma_q(x)\ \Gamma_q(y)}{\Gamma_q(x+y)}, \tag{1.1.13}$$

respectively, cf. [15]. The q-beta function has the q-integral representation

$$\beta_q(x,y) = \int_0^1 t^{x-1}\ \frac{(tq;q)_\infty}{(tq^y;q)_\infty}\ d_qt, \quad Re(x) > 0, \quad y \neq 0, -1, -2, \ldots. \tag{1.1.14}$$

1.2 q-Bessel functions

In 1905, cf. [24] Jackson introduced three Jackson q-Bessel functions of the classical Bessel functions. They are denoted by $J_\nu^{(k)}(z;q)$, $k = 1, 2, 3$, and defined by

$$J_\nu^{(1)}(z;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{(q;q)_k(q^{\nu+1};q)_k}\left(\frac{z}{2}\right)^{2k}, \ |z| < 2, \tag{1.2.1}$$

$$J_\nu^{(2)}(z;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(\nu+n)}}{(q;q)_n(q^{\nu+1};q)_n}\left(\frac{z}{2}\right)^{2n}, \ z \in \mathbb{C}, \tag{1.2.2}$$

$$J_\nu^{(3)}(z;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\ z^\nu \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)/2}}{(q;q)_n(q^{\nu+1};q)_n}\ z^{2n}, \ z \in \mathbb{C}, \tag{1.2.3}$$

respectively. The function $J_\nu^{(3)}(z;q)$ is also known in some literatures as the Hahn-Exton q-bessel function, see [24, 27, 36]. These q-analogs satisfy

$$\lim_{q\to 1^-} J_\nu^{(k)}((1-q)z;q) = J_\nu(z),\ k=1,2, \quad \lim_{q\to 1^-} J_\nu^{(3)}((1-q)z;q) = J_\nu(2z),$$

see [36]. There are several studies on zeros of $J_\nu^{(k)}(z;q)$, $k=1,2,3$. Ismail, in [20], proved that $J_\nu^{(2)}(z;q)$ has infinitely many real that interlace with those of $J_{\nu+1}^{(2)}(z;q)$ by analyzing certain orthogonal polynomials, see also [11]. The same is proved by Koelink and Swarttouw, in [28], for $J_\nu^{(3)}(z;q)$ by using q-difference equations. It is known that $z^{-\nu}J_\nu^{(k)}(z;q)$, $k=2,3$, are entire functions of order zero. Also the function $J_\nu^{(1)}(z;q)$ can be defined on $\mathbb{C}$ as

$$J_\nu^{(1)}(z;q) = \frac{J_\nu^{(2)}(z;q)}{(-z^2/4;q)_\infty},$$

cf. [35]. Thus $J_\nu^{(1)}(z;q)$ is a meromorphic function whose poles are simple and constituted the set $\left\{\pm 2q^{-j/2}i:\ \ j\in\mathbb{N}\right\}$. Ismail, cf. [20], conjectured that if $\{z_{n,\nu}\}_{n=1}^\infty$ are the positive zeros of $J_\nu^{(2)}(z;q)$, then

$$z_{n,\nu} = b_1 q^{-n/\alpha} + b_2 q^{\beta n}(1+O(q^{\gamma n})),\ \text{ as }\ n\to\infty,$$

for some constants $b_1, b_2, \alpha, \beta, \gamma$. Hayman proved in [17] that, for arbitrary k, the positive zeros $\{z_{n,\nu}\}_{n=1}^\infty$ of $J_\nu^{(2)}(z;q)$ have the following asymptotic expansion

$$z_{n,\nu} = 2q^{-n}q^{\frac{-\nu+1}{2}}\left\{1+\sum_{j=1}^{k} c_n q^{nj} + O(q^{n(k+1)})\right\},$$

for sufficiently large n, where the constants c_j, $j=1,2,\ldots,k$ depends on q and ν and can be computed iteratively. From now on, $\lambda_{n,\nu}$ will denote the nth positive zeros of $J_\nu^{(2)}(z;q^2)$, thus $\lambda_{n,\nu} = z_{n,\nu}(q^2)$.

Definition 1.2.1. The q-trigonometric functions $\mathrm{Cos}_q z$, $\mathrm{Sin}_q z$, $\cos(z;q)$ and $\sin(z;q)$ are defined on $\mathbb{C}$ by

$$\begin{aligned}
\mathrm{Cos}_q z &:= \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(2n-1)}z^{2n}}{(q;q)_{2n}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} z^{1/2} J^{(2)}_{-1/2}(2z;q^2), && (1.2.4)\\
\mathrm{Sin}_q z &:= \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(2n+1)}z^{2n+1}}{(q;q)_{2n+1}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} z^{1/2} J^{(2)}_{1/2}(2z;\, q^2), && (1.2.5)\\
\cos(z;q) &:= \sum_{n=0}^{\infty}(-1)^n \frac{q^{n^2}(z(1-q))^{2n}}{(q;q)_{2n}} && (1.2.6)\\
&= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}(zq^{-1/2}(1-q))^{1/2} J^{(3)}_{-1/2}\left(z(1-q)/\sqrt{q}\,;q^2\right), \\
\sin(z;q) &:= \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)}(z(1-q))^{2n+1}}{(q;q)_{2n+1}} && (1.2.7)\\
&= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}(z(1-q))^{1/2} J^{(3)}_{1/2}\left(z(1-q);\, q^2\right),.
\end{aligned}$$

From the above definition, it follows that the functions $\mathrm{Cos}_q z$, $\mathrm{Sin}_q z$, $\cos(z;q)$, and $\sin(z;q)$ have only real simple zeros.

Lemma 1.2.1. [9] ***i.*** *If $0 < q < \gamma_0$, where $\gamma_0 \approx 0.429052$ is the root of*

$$(1-q)(1-q^2)(1-q^3) - q, \quad q \in (0,1),$$

then the zeros of $\sin(z;q)$ are real, infinite and simple and the positive zeros are situated in the intervals

$$\left(q^{-r+1/2}\sqrt{(1-q^2)(1-q^3)},\, q^{-r-1/2}\sqrt{(1-q^2)(1-q^3)}\right), \quad r = 1,2,\ldots, \qquad (1.2.8)$$

one zero in each interval.

ii. *If $0 < q < \beta_0$, where $\beta_0 \approx 0.38197$ is the zero of $(1-q)^3(1+q) - q$ in $0 < q < 1$, then the zeros of $\cos(z;q)$ are real and simple and the positive zeros are situated in the intervals*

$$\left(q^{-r}(1-q)\sqrt{1+q},\, q^{-r-1}(1-q)\sqrt{1+q}\right), \quad r = 0,1,2,\ldots, \qquad (1.2.9)$$

one zero in each interval.

Ismail [21, p. 352] (see also [23]) defined the q-exponential function $\mathcal{E}_q(x;w)$ by

$$\mathcal{E}_q(x;w) := \frac{(w^2;q^2)_\infty}{(qw^2;q^2)_\infty}\sum_{n=0}^{\infty}(iw)^n \frac{(-ie^{i\theta}q^{(1-n)/2}, -ie^{-2i\theta}q^{(1-n)/2};q)_n}{(q;q)_n}\, q^{\frac{n^2}{4}}. \qquad (1.2.10)$$

1.3 Zeros of q-Hankel transforms

In this section, a brief account of different techniques used in previous studies for zeros of second and third Jackson q-Bessel functions and their q-Hankel transforms is given. Before doing this we introduce the following notations.
The q-moments of a function $f \in L_q^1(0,1)$ denoted by $\mu_n(f)$ is defined to be

$$\mu_n(f) := \int_0^1 t^n f(t)\, d_qt,\ \ n \in \mathbb{N}_0. \tag{1.3.1}$$

Let $\mu_{n,\nu}, \mathfrak{m}_{\nu,f}, \mathcal{M}_{\nu,f}$ denote the numbers

$$\mu_{n,\nu}(f) := \frac{\mu_{2n+2}(f)}{\mu_{2n}(f)(1-q^{2n+2+2\nu})(1-q^{2n+2})}, \tag{1.3.2}$$

$$\mathfrak{m}_{\nu,f} := \inf_{n\in\mathbb{N}_0} \mu_{n,\nu}(f), \tag{1.3.3}$$

$$\mathcal{M}_{\nu,f} := \sup_{n\in\mathbb{N}_0} \mu_{n,\nu}(f). \tag{1.3.4}$$

The existence of $\mathfrak{m}_{\nu,f}, \mathcal{M}_{\nu,f}$ is proved in [9]. The q-Hankel transforms, $\mathcal{H}_{\nu,f}^{(k)}(z;q),\ \ k = 2,3$, are defined by

$$\mathcal{H}_{\nu,f}^{(k)}(z;q) := z^{-\nu}\int_0^1 t^{-\nu} f(t) J_\nu^{(k)}(tz;q^2)\, d_qt, \quad z \in \mathbb{C},\ k = 2,3. \tag{1.3.5}$$

Now, we introduce the different approaches.

1. The Rouché and Hurwitz theorems approach. The following theorem is a Hurwitz-type theorem, it is taken from [34, p. 143].

Theorem 1.3.1. *Let $g_1(z)$, $g_2(z), \ldots, g_n(z), \ldots$ be entire functions which have real zeros only. If*

$$\lim_{n\to\infty} g_n(z) = g(z),$$

uniformly in any finite domain, the entire function $g(z)$ can have only real zeros.

By a finite domain in Theorem 1.3.1, it is meant a compact subset of $\mathbb{C}$. In this approach Annaby and Mansour [9] used Rouché's theorem and Theorem 1.3.1, to investigate zeros of $\mathcal{H}^{(3)}_{\nu,f}(z;q)$. They proved that under the condition

$$q^{-1}(1-q)\frac{\mathfrak{m}_{\nu,f}}{\mathcal{M}_{\nu,f}} > 1, \tag{1.3.6}$$

the function $\mathcal{H}^{(3)}_{\nu,f}(z;q)$ has infinitely many zeros, all of them are real, simple and lie in the intervals

$$\left(\frac{q^{-r+1/2}}{\sqrt{\mathcal{M}_{\nu,f}}}, \frac{q^{-r-1/2}}{\sqrt{\mathcal{M}_{\nu,f}}}\right), \quad r = 1, 2, 3, \ldots,$$

one zero in each interval, and it has no zeros in the interval $\left[0, q^{-1/2}/\sqrt{\mathcal{M}_{\nu,f}}\right)$. Using this technique the authors considered $\mathcal{H}^{(3)}_{\nu,f}(z;q)$ as a limit of the sequence of partial sums and applied Theorem 1.3.1 as well as Rouché's theorem.

2. The θ-function approach. This approach is concerned with the θ-function defined in (1.1.11). Bergweiler and Hayman, in [38], studied the asymptotic behavior of this function. Using their results, Annaby and Mansour proved in [11] (see also [10]) that for any $q \in (0,1)$, the function $\mathcal{H}^{(3)}_{\nu,f}(z;q)$ has at most a finite number of non-real zeros, and has an infinite number of real and simple zeros. If $\{\zeta_{n,\nu}\}_{n=1}^{\infty}$ and $\{\eta_{n,\nu}\}_{n=1}^{\infty}$ denote the sequence of positive zeros of $\mathcal{H}^{(2)}_{\nu,f}(z;q)$ and $\mathcal{H}^{(3)}_{\nu,f}(z;q)$, respectively, then (cf. [10, 11]), for sufficiently large n

$$\zeta_{n,\nu} = 2\,q^{-2n}\,q^{-\nu+1}(1+O(q^{2n})), \quad \eta_{n,\nu} = q^{-n}(1+O(q^{n})). \tag{1.3.7}$$

3. The Hurwitz–Biehler approach. The following theorem is a version of the Hurwitz-Biehler theorem for entire functions of order zero, cf. [32, chapter7].