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On odd graceful graphs and other labellings

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Summary

Summary

In this thesis. We are concerned with a very important topic in graph theory, which is graph labeling, especially with graceful, harmonious, α- labeling, odd graceful, product cordial and difference labeling. This thesis consists of four chapters.

Chapter one

In chapter one, we introduce some basic definitions and notations in graph theory which we will need afterwards.

Chapter two

In chapter two, we study the relation between α-graph and odd graceful graph, and we prove that if G is an α-graph, then $G \odot \overline{K_w}$ is odd graceful for all $w \geq 1$, if G_1 is an α-graph and G_2 is an odd graceful, then $G_1 \cup G_2$ is odd graceful. We also show that some families of graphs are α-graphs and odd graceful such as: A gear graph G_m for all $m \geq 3$ is an α-graph; the dragon is odd graceful when n is even, $n \geq 4$ and $m \geq 1$, and it is an α-graph if $n \equiv 0 \pmod 4$, $m \geq 1$; the graph $G_{m,n}$ is odd graceful when $m \geq 3$, $n \geq 1$; the graphs $C_n \odot \overline{K_w}$ are odd graceful for $w \geq 1$, n is even, $n \geq 4$; the graph $C_{n,m}^*$ is odd graceful when n is even, $n \geq 4$, $m \geq 3$ and it is α-graph if $n \equiv 0 \pmod 4$, $m \geq 3$; the butterfly graphs are odd graceful; and The graphs C_m^{**} are odd graceful when m is even and $m \geq 4$. The results of this chapter has been accepted for publication in the Canadian journal Ars Combinatoria.

Chapter three

In chapter three, We discovered some mistakes in a published paper, and showed that the claims are not correct. Also, we give some necessary conditions for a product cordial graph and we determine all graphs of order less than 7 which are not product cordial. Also ,we give some families of graphs which are product cordial and not product cordial such as: the complete graphs K_n are not product cordial for all $n \geq 4$; and the following graphs are product cordial : the cycle C_n if and only if n is odd; the gear graph G_m if and only if m is odd; the web $\overline{W_n}$ for all n; the corona $C_n \odot \overline{K_w}$ for all $w \geq 1$, $n \geq 3$; the conjunctions : $P_m \wedge P_n$ for $n,m \geq 2$; $P_n \wedge S_m$, $n,m \geq 2$; the corona graph $T_n \odot K_1$ for $n \geq 2$; the C_4 −snake if and only if n is odd ; the flower graph H_n^* and the closed Helm $\overline{H_n}$ are product cordial for all n. The results of this chapter has been accepted for publication in the Canadian journal Ars Combinatoria.

Chapter four

In chapter four, we give a survey of all graphs of order ≤ 5 which are difference graphs. Also, we show that, the graph $K_n - me$, where the m deleted edges have a common vertex; the graph $K_n - 2e$, $n \geq 4$, where the two deleted edges have no vertex in common; the graph $K_n - 3e$, $n \geq 6$, where the three deleted edges have no vertex in common are difference graphs. Also, we show that the following graphs are difference graphs: the gear graph G_n for all $n \geq 3$; the grids $P_m \times P_n$ for all n, $m \geq 3$; the triangular snakes; the C_4 − snakes; the dragon (that is, a graph formed by identifying the end vertex of the path of m edges $(m \geq 1)$ and any vertex in the cycle C_n $(n \geq 3)$); the graph C_m^{**} consists of two cycles of the same order m joined by a bridge; and $C_{n,m}$ (that is, a graph obtained by identifying the center of a star S_m with a vertex of a cycle C_n).

Chapter 1

Chapter 1
Basic Definitions and Notations

1.1 Introduction.

In this chapter, we present the basic definitions and notations in graph theory concerning α-labeling, graceful, harmonious, odd graceful, product cordial, and difference labelings as well as their known results.

The concept of graceful labeling and α-labeling were first introduced by Rosa [19] in 1967. The concept of harmonious labeling was introduced by Graham and Sloane [16] in 1980. Some recent trends in labeling of graphs, such as odd graceful, product cordial, and difference labeling will be studied.

1.2 Basic Definitions.

Let $G = (V, E)$ be a graph, where $V = V(G)$ is its vertex set and $E = E(G)$ is its edge set, the order of G is the cardinality $p = |V|$ of V, the size of G is the cardinality $q = |E|$ of E. The two vertices u and v are said to be adjacent if $uv \in E$, and if e $= uv \in E$, then we say that u and v are the end vertices of e and that the edge e is incident with u and v.

Two edges are said to be adjacent if they are incident on a common vertex. Similarly two vertices u and v in V are said to be adjacent if they are the end vertices of the same edge. The number of edges incident on a vertex v is called the degree, deg (v), of the vertex v. Isolated vertices are vertices with zero degree. A vertex of degree one is called a pendent vertex and the edge incident on it is called a pendent edge.

A regular graph is defined as a graph all of its vertices are of the same degree, in such case we say that the graph of degree d is a d-regular graph. A null graph is a regular graph of degree zero. We shall denote the null graph on n vertices by $\overline{K_n}$. A complete graph K_n is a regular graph of degree $n - 1$ where n is the number of vertices of K_n.

If the vertex set of a graph G can be partitioned into two sets V_1 and V_2 such that any edge of G joins one vertex in V_1 to one vertex in V_2 then G is called a bipartite graph having bipartition (V_1, V_2).

A bipartite graph having bipartition (V_1, V_2) is called complete if each vertex in V_1 is adjacent to every vertex in V_2. The complete graph having bipartition (V_1, V_2) such that $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. A bipartite graph of the form $K_{1,n}$ is called a star graph.

A walk of a graph is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, ..., v_{n-1}, e_n, v_n$ beginning and ending with vertices in which each edge is incident on the two vertices immediately preceding and following it, it is closed if $v_0 = v_n$ and open otherwise.

It is a trial if all edges are distinct, and a path P_n if all the vertices (and thus necessarily all the edges) are distinct, if $v_0 = v_n$, then it is called a cycle. A cycle on n vertices $n \geq 3$ is denoted by C_n. The number of edges in a path is called the length of the path.

If e is an edge of a graph G, we denote by $G - e$ the graph obtained from G by deleting the edge e. Similarly, if v is a vertex of G, we denote by $G - v$ the graph obtained from G by deleting the vertex v together with the edges incident to v [31]. Some deletions are shown in Figure 1.1.

$$G \qquad G-e \qquad G-v$$
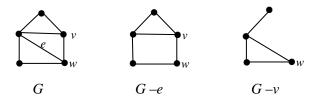
Figure 1.1.

A graph H is said to be a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We write $H \subseteq G$ if H is a subgraph of G. A subgraph H of G such that $V(H) = V(G)$, is called a spaning subgraph of G. A subgraph H of G, such that whenever $u, v \in V(H)$ are adjacent in G then they are adjacent in H is called an induced subgraph of G. A graph G is said to be connected if there is at least one path between every pair of vertices of G. Otherwise, G is disconnected.

A tree is a connected graph of order n and size $n-1$. A caterpillar is a tree with the property that the removal of its end vertices leaves a path, while a lobster is a tree with the property that the removal of the end vertices leaves a caterpillar.

Some graph operations:

Let G_1 and G_2 be two disjoint graphs. The corona $(G_1 \odot G_2)$ of G_1 and G_2 is the graph obtained by taking one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2, and then joining the jth vertex of G_1 to every vertex in the jth copy of G_2.

Let G_1 and G_2 be two disjoint graphs. The union $(G_1 \cup G_2)$ of G_1 and G_2 is the graph having vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$ [17].

Let G_1 and G_2 be two graphs. The conjunction $(G_1 \wedge G_2)$ of G_1 and G_2 is the graph having vertex set $V(G_1) \times V(G_2)$ and edge set $\{(u_1,v_1)(u_2,v_2) : u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)\}$.

The Cartesian product $G_1 \times G_2$ of G_1 and G_2 is the graph having vertex set $V(G_1) \times V(G_2)$ and edge set $\{(u_1,v_1)(u_2,v_2) : (u_1 = u_2 \text{ and } v_1v_2 \in E(G_2)) \text{ or } (v_1 = v_2 \text{ and } u_1u_2 \in E(G_1))\}$ [17].

The composition $G_1[G_2]$ of G_1 and G_2 is the graph having vertex set $V(G_1) \times V(G_2)$ and edge set $\{(u_1,v_1)(u_2,v_2) : (u_1u_2 \in E(G_1) \text{ or } (u_1 = u_2 \text{ and } v_1v_2 \in E(G_2))\}$.

The join $G_1 + G_2$ of G_1 and G_2 is the graph having vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$.

The symmetric product $G_1 \oplus G_2$ of G_1 and G_2 is the graph having vertex set $V(G_1) \times V(G_2)$ and edge set $\{(u_1, v_1)(u_2, v_2): (u_1u_2 \in E(G_1) \text{ or } v_1v_2 \in E(G_2) \text{ but not both}\}$.

The two graphs G and H are isomorphic $G \cong H$ if there is a one to one correspondence φ mapping from $V(G)$ onto $V(H)$, such that φ preserves adjacency, i.e., for every $u, v \in V(G)$, $uv \in E(G)$ if and only if $\varphi(u)\varphi(v) \in E(H)$.

1.3 Graceful Labelings.

A graph $G = (V, E)$ with $|E|$ edges is called graceful if there is an injective function f from $V(G)$ into the set $\{0, 1, 2, \ldots, |E(G)|\}$ such that the induced function f^* from $E(G)$ into the set $\{1, 2, 3, \ldots, |E(G)|\}$ defined by $f^*(uv) = |f(u) - f(v)|$ for all $uv \in E(G)$ is an injection. Rosa [19] called this labeling β-valuation. The term graceful is introduced by Golomb [14]. Rosa [19] proved that if every vertex of G has even degree and if the number of edge of G is congruent to 1 or 2(mod 4) then G is not graceful. He [19] gave a necessary condition for an Eulerian graph to be graceful, he showed that if G is a graceful Eulerian graph of m edge, then $m \equiv 0$ or 3(mod 4). The following results are due to Rosa [19]. The cycle C_n is graceful if and only if $n \equiv 0$ or 3(mod 4). The complete graph K_n is graceful if and only if $n \leq 4$. The complete bipartite graph $K_{m,n}$ is graceful for all m, n.

Ringel and Kotzig conjectured that all trees are graceful. Among the trees known to be graceful are: caterpillars, trees with at most 4 pendent vertices, trees with at most 27 vertices and olive trees (an olive tree is a rooted tree of k branches, where the ith branch is a path of length i).

Many authors have established this conjecture for special cases. Truszczynski studied unicyclic graphs and proved several classes of such graphs are graceful. Among these he calls dragons (a dragon is formed by joining the end vertex of a path to a cycle). This led Truszczynski to conjecture that all unicyclic connected graphs except C_n, where $n \equiv 1$ or 2(mod 4) are graceful (see Gallian [4]).

Graphs that are Cartesian products have been the subject of many papers. The grids $P_m \times P_n$ are graceful. This had been proved by Acharya and Gill in 1978. Prisms are graphs of the form $C_n \times P_n$. In 1977 Bodendiek, Schumacher and Wegner proved that $C_m \times P_2$ is graceful when $m \equiv 0$ (mod 4). According to the survey by Bermond [2], Gangopadhyay and Rao Hebbare did the case that m is even about the same time. In 1992 Jungreis and Reid [18] proved that all $C_m \times P_n$ are graceful when m and n are even or when $m \equiv 0$ (mod 4). Singh proved that $C_3 \times P_n$ is graceful for all n. Huang and Skiena have shown that $C_m \times P_n$ is graceful for all n when m is even and for all n with $3 \leq n \leq 12$ when m is odd. Torus grids are graphs of the form $C_m \times C_n$, $m, n \geq 12$. Reid [18] showed that $C_m \times C_n$ is graceful when $m \equiv 0$ (mod 4) and n is even. In 1980 Maheo proved that the books of the form B_{2m} are graceful and conjectured that the books B_{4m+1} are also graceful.

In 1988 Gallian and Jungreis [10] proved that the stacked books of the form $S_{2m} \times P_n$ are graceful. Gallian [7] showed that all Mobius ladders M_n are graceful. The wheel W_n was proved to be graceful by Frucht, Hoede and Kuiper. Helms were proved to be graceful by Ayle and Favaron. The windmill graph $K_n^{(m)}, n \geq 3$ is defined to be the graph consisting of m copies of K_n with one vertex in common.

Bermond [2] has conjectured that $K_4^{(m)}$ is graceful for all $m \geq 4$. This is known to be true for $4 \leq m \leq 22$. Kohetal introduced the notation $B(n, r, m)$ for the graph consisting of m copies of K_n with an K_r in common. Bermond [2] raised the question: " For which m, n and r is $B(n, r, m)$ graceful?''. Of Cours the case $r = 1$ is the same as $K_n^{(m)}$. For $r > 1$, Koh, Rogers and Lim proved that $B(n, r, m)$ is graceful in the following cases: $n = 3, r = 2, m \geq 1$; $n = 4, r = 3, m \geq 1$, while Delorme proved that $B(n, r, m)$ is graceful when $n = 4, r = 2$ and $m \geq 1$ (see Gallian [4]).

Seoud and Youssef [24] proved that $K_{1,p,q}$, $K_{2,2,p}$, $K_{1,1,p,q}$ are graceful for all p and q, also they proved in [26] that : the join of any two stars is graceful, the join of any path and any stars is graceful and generally, the join of any graceful tree and star is graceful.

1.4 Harmonious Labelings.

Harmonious graphs naturally arose in the study by Graham and Sloane [16]. They defined a graph G with q edges to be harmonious if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge uv is assigned the label $f(u) + f(v)$ (mod q), the resulting edge labels are distinct.

They proved that if a harmonious graph G has an even size q and the degree of every vertex is divisible by 2^k then q is divisible by 2^{k+1}.

They proved the following results : K_n is harmonious if and only if $n \leq 4$, $K_{m,n}$ is harmonious if m or $n = 1$, all trees of order $= 9$ are harmonious. C_n, $n \geq 3$ is harmonious if and only if n is odd, all wheels W_n $n \geq 3$, fans f_n, $n \geq 2$ and double fans g_n, $n \geq 2$ are harmonious ; the friendship graph is harmonious if and only if $n \neq 2$ (mod 4); all caterpillars are harmonious; all ladders L_n except for $n = 2$ are harmonious; $C_m \times P_n$ is harmonious when m is odd, and they used a computer to show that $C_4 \times P_2$ is not harmonious.

Gallian [7] showed that all Mobius ladders M_n except for $n = 3$ are harmonious. In 1992 Gallian, Prout and Winters [11] proved that: $C_m \times P_2$ is harmonious when $m \neq 4$ and all prisms $C_m \times P_2$ with a single vertex deleted or a single edge deleted are harmonious.

In 1992 Jungreis and Reid [18] showed that; all grids $P_m \times P_n$ are harmonious when $(m, n) \neq (2,2)$; $C_4 \times P_n$ is harmonious when $n \geq 3$; $C_{4m} \times P_n$ is harmonious when $(m, n) \neq (2,2)$ and $C_{4m+2} \times P_{2n}$ is harmonious when $n > 1$.

Aldred and Meckay used a computer to show that all trees with at most 26 vertices are harmonious. Grace [15] proved that an odd cycle with one or more pendent edges at each vertex is harmonious and conjectured that an even cycle with one pendent edge attached at each vertex is harmonious. This conjecture has been proved by Liu and Zhang and others. Reid proved that $K_4 \times P_n$ is harmonious for all m.

Hsu proved that $K_4^{(m)}$ is harmonious for all m (see Gallian [7]). Seoud, Abdel Maqsoud and Sheehan [21] proved that: $C_n \cup C_{n+1}$ is harmonious for all $n \leq 4$; P_n^3 is harmonious for all n. They conjectured that P_n^k, $n \geq k + 1$ is harmonious if and only if $k \leq 3$. Seoud and Youssef [24] proved that $K_{m,n,p}$ is not harmonious when m, n and p are congruent to 2 modulo 4 and the graphs $B(3,2,m)$ and $B(4,3,m)$ are harmonious for all m; $C_n + \overline{K_p}$ is harmonious for every p when n is odd and if any edge is added to $K_{m,n}$, and $P_n[P_2]$ is harmonious.

1.5 α – Labelings.

In [19] Rosa defined an α –labeling to be a graceful labeling with the additional property that there exists an integer k so that for each edge uv either $f(u) \leq k \leq f(v)$ or $f(v) \leq k \leq f(u)$. Fig. 1.2 proves that the path P_8 is α –labeling.

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Figure. 1.2

It follows that such a k must be the smaller of the two vertex labels that yield the edge labeled 1. Also, a graph with an α –labeling is necessarily bipartite and therefore cannot contain a cycle of odd length. Graphs with α –labeling proved useful in the development of the theory of graph decompositions.