Ain Shams University
Faculty of Education
Department of Mathematics

Symmetries, Exact Solutions and their Applications to Nonlinear Partial Differential Equations of Interesting Physical Problems

A Thesis

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Summary

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SUMMARY

The objective of this thesis is to find exact solutions of partial differential equations which appear in a wide variety of physical applications. This thesis consists of six chapters organized as follows:

Chapter (I): In this chapter, the attention is confined to give survey and development of the various techniques utilized in chapters II-VI to obtain exact solutions of nonlinear evolution equations.

Chapter (II): In this chapter we have solved two systems and one single equation of nonlinear partial differential equations which appear in a wide variety of physical applications. More specifically by using infinitesimal symmetries, we found four vectors field in the first system, eight vectors in the second system and seven vectors in the last equations. By these vectors we reduced the first system to only two ordinary differential equations.

The second section has been accepted in Mathematical method in applied science.

Chapter (III): In this chapter we have taken up the application of the Lie group method by using the extension of the infinitesimal operator (Prolongation) to investigate invariant solutions of nonlinear Boltzman equation, b-family equations and boundary layer stagnation-point flow towards a heated porous stretching sheet.

The second section has been published in American Journal of Mathematics and Statistics.

Chapter (IV): Herein, we study the integrability of nonlinear partial differential equations, we obtained new exact solution via Painlevé property and Bäckland transformation. We have written the Painlevé expansion about the moving singular manifold and the Bäckland transformation, then we have supposed a wave form to the manifold function involved in the expansion

The second section has been published in American Journal of Computational and Applied Mathematics.

Chapter (V): We make use of Auxiliary equation method, The Generalized He's Exp-function method and new extended Fractional-Expansion method for finding a new exact solutions of Non-Linear Klein-Gordon equation, Drinfel'd-Sokolov-Wilson system, (2 + 1)-dimensional

variable coefficient Broer-Kaup system and (3+1)-dimensional Burger equation.

The first section has been published in Applied Mathematics.

<u>Chapter (VI)</u>: The objective of this chapter is to study the exact solutions and analytical solutions of the nonlinear partial differential equations by applying Homotopy Perturbation method.

In the first section we obtained analytical solutions of Complexly coupled KdV system. The second section we applied Homotopy Perturbation method for the generalized coupled mKdV equation with multicomponent. The solutions introduced in this study are in recursive sequence forms which can be used to obtain the closed form of the solutions if they are required.

The first section has been accepted in ISRN Applied Mathematics.

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Chapter (II): In this chapter we have solved two systems and one single equation of nonlinear partial differential equations which appear in a wide variety of physical applications. More specifically by using infinitesimal symmetries, we found four vectors field in the first system, eight vectors in the second system and seven vectors in the last equations. By these vectors we reduced the first system to only two ordinary differential equations. The second system has been reduced to nonlinear system of partial differential equations which have been reduced to ordinary differential equations via transformations. The last equation also has been reduced to partial differential equations which have been reduced to ordinary differential equations via the Scaling method and transformations. In the final we solved these ordinary differential equations and obtained a new exact solutions for the (1+1)-dimensional complexly KdV, the (2+1)-dimensional Burger system and (3+1)-dimensional Burger equation.

Chapter (III): In this chapter we have taken up the application of the Lie group method by using the extension of the infinitesimal operator (Prolongation) to investigate invariant solutions of nonlinear Boltzman equation, b-family equations and boundary layer stagnation-point flow towards a heated porous stretching sheet. The solution of the over determined partial differential equations generated from the

invariant condition, has led us to three cases for determination of the infinitesimals in the first, second equations and four in the last system. The search for solutions to the surface condition in each case has yielded a reduction to ordinary differential equations then to exact solutions.

The second section has been published in American Journal of Mathematics and Statistics.

Chapter (IV): Herein, we study the integrability of nonlinear partial differential equations, we obtained new exact solution via Painlevé property and Bäckland transformation. We have written the Painlevé expansion about the moving singular manifold and the Bäckland transformation, then we have supposed a wave form to the manifold function involved in the expansion. The substitution of this in given differential equation, this procedure has yielded exact solutions to the (3+1)-dimensional Burger equation and complexly coupled KdV system.

The second section has been published in American Journal of Computational and Applied Mathematics.

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The first section has been published in Applied Mathematics.

Chapter (VI): The objective of this chapter is to study the exact solutions and analytical solutions of the nonlinear partial differential equations by applying Homotopy Perturbation method.

In the first section we obtained analytical solutions of Complexly coupled KdV system, then we made comparison between the exact and approximate solution led us that the Homotopy Perturbation method is a powerful method and the solutions

obtained are so identical with the exact solutions.

The second section we applied Homotopy Perturbation method for the generalized coupled mKdV equation with multicomponent. The solutions introduced in this study are in recursive sequence forms which can be used to obtain the closed form of the solutions if they are required. The method is tested on various examples which are revealing the effectiveness and the simplicity of the method..

CHAPTER I

INTRODUCTION, SURVEY OF LITERATURE AND MATHEMATICAL TOOLS

1.1 General Introduction

In recent years there have been considerable developments in symmetry methods (group method) for differential equations as evidenced by the number of research paper devoted to the subject [57, 64]. This no doubt due to the basic applicability of the methods to nonlinear differential equations, symmetry methods for differential equations, originally developed by Sophees Lie, are highly algorithmic.

Lie groups, and hence their infinitesimal generators, can be naturally extended or prolonged to act on the space of independent variables, dependent variables and derivatives of the dependent variable up to any finite order. As a consequence, the seemingly intractable nonlinear conditions of the group invariance of a given system of differential equations reduce to linear homogeneous partial differential equations determining the infinitesimal generators of the group. Since the determining equations form an over determined system of linear homogeneous partial differential equation, one can usually determine the infinitesimal generators in closed form.

If a system of partial differential equations is invariant under a Lie group of point transformations, one can find, constructively, special solutions, called similarity solutions or invariant solutions, which are invariant under some subgroup of the full group admitted by the system. These solutions result from solving a reduced system of differential equations with fewer independent variables. This application of Lie groups was discovered by Lie but first came to prominence in late 1950s through the work of the soviet group at Novosibirsk, published by Ovsiannikov [70]and was subsequently translated by Bluman (1967). Invariant solutions can also be constructed for specific boundary value problems. Here one seek a group of full group of a given partial differential equation which leaves boundary curves and condition imposed on them invariant [cf. Bluman and Cole] [5]. Ibragimov [39] is concerned with applications of mathematical physics, geometry and mechanics given by stinger and Weaver in (1986). An excellent general theory of the subject is presented by Olver [69].

Bhutani et all in (1986) enlarged this technique to deal with systems of partial differential equations then the technique has found an important place in the literature of group theoretic methods see [4, 5, 18, 49, 59 − 64] also recently Singh K. et all [17] applied this symmetry method to variable coefficients systems namely generalized Hirota-Satsuma coupled KdV system and variant Boussinesq system. Using the non-equivalent Lie analysis for each essential vector field, the nonlinear systems reduced to systems of ordinary differential equations (ODEs), and some exact solutions of those systems are found.

The investigation of the travelling wave solutions for non-linear partial differential equations plays an important role in the study of non-linear physical phenomena. Non-linear wave phenomena appears in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and geochemistry. Non-linear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in non-linear wave equations. In recent years, new exact solutions may help to find new phenomena. A variety of powerful methods, such as inverse scattering method [76], bilinear transformation (Hirota 1980), the tanh-sech method [54,55,78], extended tanh method [27,28], sine-cosine method [81], homogeneous balance method [28], Exp-function method [37, 38] and Generalization of He's Exp-function Method [25] method were used to develop non-linear dispersive and dissipative problems.

After given a brief survey of the available literature related to the research problems put up in chapter II-IV we outline in the following section, for ready reference, certain characteristic features of the methods utilized.

1.2 Symmetry method

We briefly outlined Steinberg's [73] similarity method of finding explicit solutions of both linear and non-linear partial differential equations. The method based on finding the symmetries of the differential equation as follows:

Suppose that the differential operator L can be written in the form

$$L(u) = \frac{\partial^p u}{\partial t^p} - H(u), \tag{1.2.1}$$

where $u = u(t, x)$ and H may depend on t, x, u and any derivative of u as long the derivative of u dose not contain more than $(p-1)$ t derivatives. Consider the symmetry operator called infinitesimal symmetry, which being quasi-linear partial differential operator of first-order, has the form

$$S(u) = A(t, x, u)\frac{\partial u}{\partial t} + \sum_{i=1}^{n} B_i(t, x, u)\frac{\partial u}{\partial x_i} + C(t, x, u). \tag{1.2.2}$$

Define the Frèchet derivative of $L(u)$ by

$$F(L, u, v) = \frac{d}{d\varepsilon} L(u + \varepsilon v)|_{\varepsilon=0}. \tag{1.2.3}$$

With these definitions in mind we need to follow the following steps:

(i) Compute$F(L, u, v)$,

(ii) Compute $F(L, u, S(u))$,

(iii) Substitute $H(u)$ for $(\frac{\partial^p u}{\partial t^p})$ in $F(L, u, S(u))$,

(iv) Set this expression to zero and perform a polynomial expansion.