Faculty of Education
Mathematics Department

Analytical Solutions for Some Biomechanical Problems

Thesis

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(Applied Mathematics)

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Summary

In this thesis, we studied and solved analytically some problems of electrohydrodynamic peristaltic flow with heat transfer in the presence of ac electric field. The stream function, temperature distribution and electric potential function are obtained up to second order in terms of electrical Rayleigh number, temperature parameter, Reynolds number, amplitude ratio, wave number and channel width. In our analysis, we assumed that the velocity components, the pressure gradient, the temperature and the electric potential could be expanded in a regular perturbation series of the amplitude ratio.

The present thesis consists of three chapters in addition to two appendices and references section and arabic and english summaries. These chapters are outlined as follows:

Chapter 1

Chapter 1 is an introduction. We give in it some information about the following items:

- Fluid mechanics.
- Stress and Strain.
- Newtonian's law of viscosity.
- On non-Newtonian fluids.
- Biomechanics.
- On perturbation theory.
- Heat transfer.

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Chapter 1

Introduction

1.1 Fluid mechanics

The matter is usually divided into two classes: solids and fluids. Solid is the portion of matter which has a definite shape under given thermodynamic conditions and in the absence of external forces. A fluid is a substance which cannot resist a shear force or stress without moving as a solid. Fluids are usually classified as liquids, gases and plasmas. A liquid has intermolecular forces which hold together so that it possesses volume but no definite shape. A liquid poured will fill the container up to the volume of the liquid regardless of the container's shape. Liquids have slight compressibility and the density varies little with temperature or pressure. For most purposes it is sufficient to regard liquids as "incompressible fluids". A gas consists of molecules in motion which collide with each other tending to disperse it so that a gas has no set volume or shape. A gas will fill any container into which it is placed and therefore known as a "compressible fluid". Plasmas is considered to be a state of matter where the molecules are ionized fully or partially and it does not also have a definite volume as gases [[3] and [39]].

Fluid mechanics can be divided into fluid statics (the study of fluids at rest), fluid kinematics (the study of fluids in motion) and fluid dynamics (the study of the effect of forces on fluid motion). The fluid mechanics is a branch of continuum mechanics (as shown in figure 1.1).

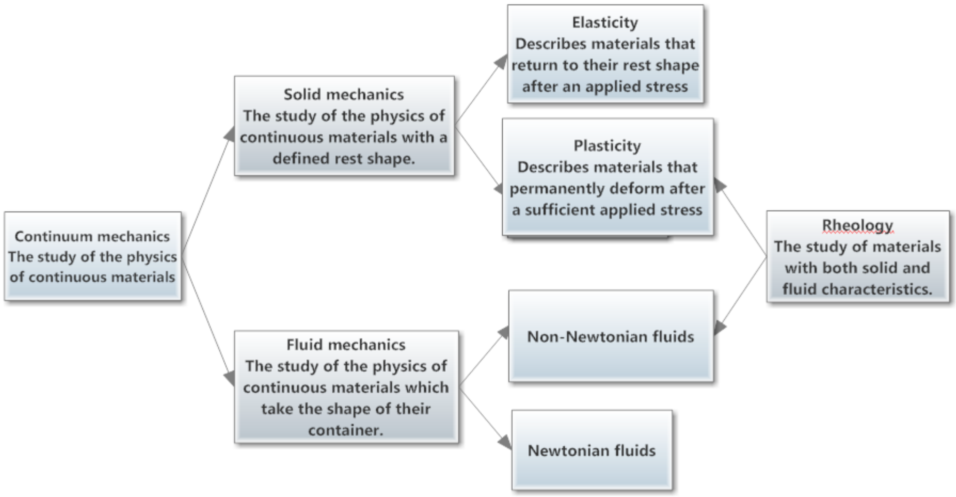

Figure 1.1: Relationship between fluid mechanics and continuum mechanics.

1.2 Stress and Strain

Stress is a measure of the internal reaction between elementary particles of a material in resisting separation, compaction, or sliding that tends to be induced by external forces. Total internal resisting forces are resultants of continuously distributed normal and parallel forces. These forces are of varying magnitude and direction and are acting on elementary areas throughout the material. These forces may be distributed uniformly or non-uniformly. Stresses are identified as tensile, compressive, or shearing, according to the straining action. Strain is a measure of deformation such as:

1. Linear strain, the change of length per unit of linear dimensions.
2. Shear strain, the angular rotation in radians of an element undergoing change of shape by shearing forces.
3. Volumetric strain, the change of volume per unit of volume.

The strains associated with stress are characteristic of the material. Strains completely recoverable on removal of stress are called elastic strains. Above a critical stress, both elastic and plastic strains exist. The part remaining after unloading represents plastic deformation called inelastic strain. Inelastic strain reflects internal changes in the crystalline structure of the metal. Shear strain is a strain that acts parallel to the surface of a material. Normal strain acts perpendicular to the surface of that it is acting on. There are two ways to interpret shear strain: the average shear strain and the engineering shear strain [89].

The following assumptions illustrate the relationship connecting the stress and rate of strain components (Stokes assumptions) [3]:

1. The stress components are linear functions of the rate of strain components (Newtonian fluid).

2. The relations between stress components and rate of strain components are invariant to orientation of the coordinate axes, i.e., they remain unchanged by a rotation of the system of coordinates or by interchange of axes (Isotropic fluid).

3. When all the velocity gradients are zero the stress components must reduce to hydrostatic pressure.

1.3 Newtonian's law of viscosity

Viscosity of a fluid is that characteristic of real fluids which exhibits a certain resistance to alterations of form. Viscosity is also known as internal friction. All known fluids possess the property of viscosity in varying degrees. According to the Newtonian's law of viscosity, the shear stress on a fluid elemental layer is directly proportional to the rate of strain. The constant of proportionality between them is known as the coefficient of viscosity [89].
A simple equation to describe Newtonian fluid behavior is

$$\tau = -\mu \frac{dv}{dy}, \tag{1.1}$$

where τ is the shear stress exerted by the fluid, μ is the coefficient of viscosity (a constant of proportionality), dv/dy is the velocity gradient

perpendicular to the direction of shear (rate of strain).
When a fluid is sheared between a fixed plate and a moving plate, the coefficient of viscosity is given by the Eq.

$$\mu = \frac{shearingstress}{velocitygradient} = \frac{force/area}{velocity/height}. \tag{1.2}$$

The table 1.1 lists the coefficient of viscosity of some materials.
Thus, as the viscosity of a fluid increases, it requires a larger force to move the top plate at a given velocity. For a Newtonian fluid, the viscosity depends only on temperature and pressure and does not depend on the forces acting upon it. But for non-Newtonian fluids, the viscosity can change by many orders of magnitude as the shear rate (velocity/height) changes (see Fig. 1.2).

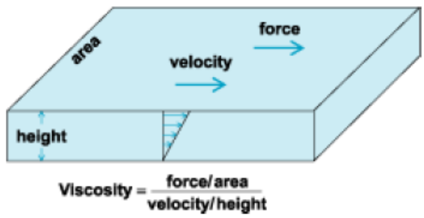

Figure 1.2: Flow between two parallel plates to illustrate viscosity.

The effect of viscosity on the motion of a fluid is determined by the ratio of μ to the density ρ. This ratio is known as "kinematic viscosity". If a fluid obeys this relation (the relation between stress and strain is linear), it is termed a Newtonian fluid. If a fluid does not obey this relation, it is termed a non-Newtonian fluid [7].

Many of the fluids encountered in everyday life (such as water, air, gasoline, mineral oils, molten metals and honey) are adequately described as being Newtonian fluids, but (food stuffs (fruit/vegetable purees and concentrates, sauces, salad dressings, mayonnaise, jams and marmalades, ice cream, soups, cake mixes and cake toppings, egg white, bread mixes, snacks and ketchup), greases and lubricating oils, molten lava and magmas, polymer melts and solutions, reinforced plastics, rubber, custard, mine tailings and mineral suspensions, paint, polishes, varnishes, paper pulp suspensions, multi-grade engine oils, peat and lignite slurries, adhesives (wall paper paste and carpet adhesive), ales (beer and liqueurs), animal waste slurries from cattle farms, biological fluids (blood, synovial fluid and saliva), bitumen, cement paste and slurries, chalk slurries, chocolates, coal slurries, cosmetics and personal care products (nail polish, lotions and creams, lipsticks, shampoos, shaving foams and creams, toothpaste, liquid soaps, etc), dairy products and dairy waste streams (cheese, butter, yogurt, fresh cream, whey, for instance), drilling mud, fire fighting foams, printing colors, inks, pharmaceutical products (creams, foams, suspensions, for instance)) are non-Newtonian fluids [4].

Table 1.1: Values of μ for common fluids at 15^0C and under atmospheric pressure.

Substance	μ (c.g.s.)	*Substance*	μ (c.g.s.)
Air	0.00018	*Olive oil*	0.1
Nitrogen	0.00017	*Paraffin oil*	0.2
Oxygen	0.0002	*Honey*	10
Hydrogen	0.00009	*Glycerine*	13
Helium	0.0002	*Caster oil*	15
Carbon dioxid	0.00014	*Corn syrup*	100
Water	0.0114	*Bitumen*	10^8
Ethyl alcohol	0.0012	*Ethylene glycol*	0.02
Mercury	0.0016	*Molten glass*	10^{12}

1.4 On non Newtonian fluids

Non-Newtonian fluids can be classified as time independent fluids, time dependent fluids and viscoelastic fluids. We summarize them briefly as follows [[7] and [8]]:

1.4.1 Time independent fluids

The viscosity of time independent fluid is dependent on the shear rate. It has the three following types (as shown in figure 1.3):

1. **Shear thickening (dilatant) fluids**
   The apparent viscosity increases with increased shear rate. Some examples of shear thickening fluids are sugar in water, clay slurries and corn starch. Shear thickening fluids are also used in all wheel drive systems utili-sing a viscous coupling unit for power transmission.

2. **Shear thinning (pseudo-plastic) fluids**
   The apparent viscosity decreases with increased shear rate. Some examples of shear thinning are paper pulp in water, milk, gelatin, polymer, paint (one wants the paint to flow readily off the brush when it is being applied to the surface being painted, but not to drip excessively), ice, blood, syrup and molasses.

3. **Visco-plastic fluids**
   This type of non-Newtonian fluid behavior is characterized by the existence of a threshold stress (called yield stress) which must be exceeded for the fluid to deform (shear) or flow. Conversely, such a substance will behave like an elastic solid or like a rigid body when the externally applied stress is less than the yield stress.

   Bingham plastics are a special class of Visco-plastic fluids which have a linear shear stress or shear strain relationship which require a finite yield stress before they begin to flow (the plot of shear stress against shear strain does not pass through the origin). Several examples are clay suspensions, drilling mud, toothpaste, mayonnaise, chocolate, and mustard. The surface of a Bingham plastic can hold peaks when it is still while a Newtonian fluids have flat featureless surfaces.

4. **Generalized Newtonian fluids**
   Stress is directly proportional to rate of strain. Some examples are blood plasma and custard.

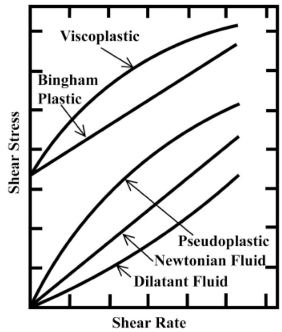

Figure 1.3: Flow curves of time independent fluids.

1.4.2 Time-dependent fluids

It has the two following types (as shown in figure 1.4):

1. **Rheopectic**
   Apparent viscosity increases with duration of shear rate. Rheopectic fluids are rare. Some examples are lubricants, whipped cream, pastes and printers inks.

2. **Thixotropic**
   Apparent viscosity decreases with duration of shear rate. Some examples are clays, some drilling mud, many paints and synovial.

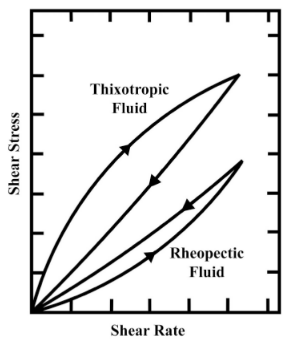

Figure 1.4: Flow curves of time dependent fluids.

1.4.3 Viscoelastic fluids

In this type of fluids both viscous and elastic properties have been possessing. The simplest type of such a material is one which is Newtonian in viscosity and obeys Hooks law for the plastic part. Viscoelastic fluids have not a simple relationships between shear stress and shear rate, but it depend on the time derivatives of both of these quantities. Some examples are polymer melts, bread dough and egg white.

1.4.4 The constitutive equations for some models of non Newtonian fluids

- **Time independent fluids**[73]

  1. **Power-law (Ostwald-deWaele)**:
     The constitutive equation for this model is

     $$\sigma = K\dot{\gamma}^n, \tag{1.3}$$

     where K and $\dot{\gamma}^n$ are the consistency index for the non Newtonian viscosity and shear rate respectively. The exponent n delineates three cases:
     $n < 1$ Pseudo-plastic fluid.
     $n = 1$ Newtonian fluid (K=μ).
     $n > 1$ Diclant fluid.

  2. **The Bingham plastic**:
     The constitutive equation for this model is

     $$\begin{aligned} &\tau = \tau_B + \mu_{PI}\dot{\gamma}, && \mid \tau \mid \geq \tau_B \\ &\dot{\gamma} = 0, && \mid \tau \mid < \tau_B \end{aligned} \tag{1.4}$$

     where τ is the extra stress tensor, τ_B is the yield stress and μ_{PI} is the plastic viscosity.

  3. **Herschel-Bulkley**:
     The Herschel-Bulkley model describes blood behavior. Some examples of fluids behaving in this manner include food products, pharmaceutical products, slurries and semisolid materials. The constitutive equation for this model is

     $$\sigma = -PI + 2(\mu + \eta)D, \tag{1.5}$$

where σ is the cauchy stress tensor, D=$\frac{1}{2}(L+L^T)$ is the symmetric part of the velocity gradient, $L=\nabla V$, T is the transpose, -PI denotes the indeterminate part of the stress due to the constraint of incompressible, μ and η are viscosities.

- **Time dependent fluids**[89]

1. **Johnson-Segalman model**:

$$\sigma=-PI+2\eta D+S, \tag{1.6}$$

$$S+\lambda\left(\frac{dS}{dt}+S(W-\alpha D)+(W-\alpha D)^T S\right)=2\mu D, \tag{1.7}$$

Where W=$\frac{1}{2}(L-L^T)$ is the skew symmetric parts of the velocity gradient, D=$\frac{1}{2}(L+L^T)$ is the symmetric part of the velocity gradient, $L=\nabla V$, λ is the relaxation time, S is the extra stress tensor and α is the slip parameter. It should be noted that this model includes the viscous Navier-Stokes fluid as a special case for $\lambda=0$. Further, when $\alpha=1$, the Johnson-Segalman model reduces to the Oldroyd-B fluid; and when $\alpha=1$ and $\eta=0$, the model reduces to the Maxwell fluid.

2. **Upper convection Maxwell model**:

$$\sigma=-PI+S, \tag{1.8}$$

$$S+\lambda\frac{DS}{Dt}=\mu A_1, \tag{1.9}$$

where $A_1=L+L^T$ is the first Rivlin-Ericksen tensor, $L=\nabla V$ and $\frac{DS}{Dt}$ is the upper convected time derivative of the stress tensor, defined by

$$\frac{DS}{Dt}=\frac{\partial S}{\partial t}+(V.\nabla)S-LS-SL^T. \tag{1.10}$$

This model becomes Newtonian when $\lambda=0$.

3. **Oldroyd-B model**:

$$\sigma=-PI+S, \tag{1.11}$$