On Various Structures of Dynamic Equations with Applications to Predator-prey Models

Presented by

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Abstract

Different structures of dynamic equation modeling, time-delay, q-difference and fractional equations are considered. Attention is focused to analyze the stability of equilibrium states in prey-predator systems with self or cross diffusion between two capita. In these modeling equations, the main objective is to determine a subspace of the parameter space, where Turing instability occurs. Further conditions of consistency of each modeling equation are constructed. The formal exact solutions of the linearized equation of the q-prey-predator model and the fraction prey-predator were obtained. The geometric structure of trajectories near equilibrium points was visualized. Some new phenomena are found to occur. Spontaneous oscillations in solutions of q-dynamic equation do produce at a specific time value.

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0.1 Introduction

Biological systems exhibit a vast varieties of phenomena and environmental conditions. So that modeling of real system is not a banal work. To have been able to interpret the different phenomena, and to meat the environmental conditions, sophisticated mathematical models are needed. In the beginning of the last century, modeling for dynamical systems were constructed via semilinear instantaneous ordinary or partial differential equations. These models could have been accepted to fit partially or completely with real phenomena. The objective of this thesis is to study prey-predator model with self or cross diffusion. The analysis is done for many different structures of model equations. The formulation of prey-predator in two capita with self diffusion, in the ordinary derivative, was recently presented in [1,2].

$$
\begin{aligned}
\frac{du_1(t)}{dt} &= \varepsilon u_1(t)(1-\frac{u_1(t)}{k}) - \frac{\beta u_1(t)v_1(t)}{\beta+u_1(t)} + d_1(\rho(u_2(t))-\rho(u_1(t))),\\
\frac{dv_1(t)}{dt} &= -\frac{v_1(t)(\gamma+\delta v_1(t))}{1+v_1} + \frac{\beta u_1(t)v_1(t)}{\beta+u_1(t)} + d_2(\rho(v_2(t))-\rho(v_1(t))),\\
\frac{du_2(t)}{dt} &= \varepsilon u_2(t)(1-\frac{u_2(t)}{k}) - \frac{\beta u_2(t)v_2(t)}{\beta+u_2(t)} + d_1(\rho(u_1(t))-\rho(u_2(t))),\\
\frac{dv_2(t)}{dt} &= -\frac{v_2(t)(\gamma+\delta v_2(t))}{1+v_2(t)} + \frac{\beta u_2(t)v_2(t)}{\beta+u_2(t)} + d_2(\rho(v_1(t))-\rho(v_2(t))),
\end{aligned}
\tag{0.1.1}
$$

where $u_i(t) = u(t,i)$, are the density of preys in patches i at time t , $i = 1,2: \ t \in R^+$. and $v_i(t) = v(t,i)$, are the density of predators in patches i at time t , $i = 1,2: \ t \in R^+$, $\varepsilon > 0$ is the specific growth rate of the prey in the absence of predation in both species and without environmental limitation, $\beta > 0, \ k > 0$ are the conversion rate and carrying capacity with respect to the prey, respectively, $\gamma > 0$ and $\delta > 0$, are the minimal mortality and the limiting mortality of the predator, respectively (the natural assumption is $\gamma < \delta$). The meaning of the half saturation constant is that at $u_i = \beta$, $(i = 1,2)$, the specific growth rate $\frac{\beta u_i}{\beta+u_i}$, $(i = 1,2)$ (called also a Holling type functional response) of the predator is equal to half its maximum β (the conversion rate is taken to be equal to the half saturation constant for sake of simplicity). $d_i > 0$, $(i = 1,2)$ are the diffusion coefficients.

The following conditions are considered natural;

$$
\begin{gathered}
\gamma < \beta < \delta \ , \ \beta < k,\\
and\\
\gamma < \frac{\beta k}{\beta-k}.
\end{gathered}
\tag{0.1.2}
$$

For the cross diffusion model we set $\rho(u_i) \rightarrow \rho(v_i)$ and $\rho(v_i) \rightarrow \rho(u_i)$ in equation (0.1.1). The main aim of studying the equation (0.1.1) was to find conditions for Turing instability to occur due to self or cross diffusion. To this end, it was assumed that $\rho(u_i) = d_1 u_i$ and $\rho(v_i) = d_2 v_i$. Thus the problem was reduced to find the region Ω in the plane $d_1 d_2$ so that for d_1 and d_2 in Ω, by assuming that all the other parameters are fixed, so that Turing instability holds. It is worth noticing that when $d_1 = d_2 = 0$, in the equation (0.1.1) , then it describes a system without diffusion.

One of the different equation modeling structures, based on incorporating effects of translational time delay (or locally history), was proposed to have been able to explain some features of biological systems. Translational time delay effects on the stability of biological systems had been studied, only for two or three dimensions, in the literature (see [3-5]). To our knowledge, higher dimensional prey-predator systems with time delay were not worked up. On the other hand the point about the existence of a threshold value for τ for the model equation to hold was not addressed. The criteria for justifying the use of a specifics model equation will be treated later on.

The main new topics in this thesis are as follows:

0.1.1 Time delay equation modelling

We consider the prey-predator model with self diffusion when $\rho(u_i) = d_1 u_i$ and $\rho(v_i) = d_2 v_i$ and with time delay in the diffused predators, for which we think it is the first time to consider. So that the equation (0.1.1) becomes

$$\begin{aligned}
\frac{du_1(t)}{dt} &= \varepsilon u_1(t)(1 - \frac{u_1(t)}{k}) - \frac{\beta u_1(t) v_1(t)}{\beta + u_1(t)} + d_1(\rho(u_2(t)) - \rho(u_1(t))), \\
\frac{dv_1(t)}{dt} &= -\frac{v_1(t)(\gamma + \delta v_1(t))}{1 + v_1} + \frac{\beta u_1(t) v_1(t)}{\beta + u_1(t)} + d_2(\rho(v_2(t-\tau)) - \rho(v_1(t-\tau))), \\
\frac{du_2(t)}{dt} &= \varepsilon u_2(t)(1 - \frac{u_2(t)}{k}) - \frac{\beta u_2(t) v_2(t)}{\beta + u_2(t)} + d_1(\rho(u_1(t)) - \rho(u_2(t))), \\
\frac{dv_2(t)}{dt} &= -\frac{v_2(t)(\gamma + \delta v_2(t))}{1 + v_2(t)} + \frac{\beta u_2(t) v_2(t)}{\beta + u_2(t)} + d_2(\rho(v_1(t-\tau)) - \rho(v_2(t-\tau))).
\end{aligned} \tag{0.1.3}$$

The translational time delay in (0.1.3) may be attributed to taking into consideration of the maturity effects of the species that "migrate" between the two communities.

Translationaltime delay modeling revealed many interesting aspects. If a system is asymptotically stable in the absence of delay, a variety of situations do occur in the presence of time delay. This system may be asymptotically stable (oscillatory or not). Or it has a limit cycle solution, many limit cycle solutions that may lead to chaos to occur. However the conditions that time-delay "τ" model equations are relevant were not discussed.

We think that, to our knowledge the equation (0.1.3) was not considered in the literature.

The main objective here is determine a region Ω_τ in the space $d_1 d_2 \tau$ where Turing instability occurs. On the otherhand, the existence of limit cycle solution is analyzed.

To put in evidence the validity of the model equation (0.1.3) that is, by inspecting the threshold value for τ, namely τ_c, can be done by presenting a proof for the existence of solution to (0.1.3). But this is not too simple. Even for a single equation, finding τ_c from a theoretical proof is not evident. Usually in the proofs, it is assumed that τ is sufficently small so that perturbation is allowable so that the proof is carried out [6].

However, Inspection of τ_c can be found by intuition. With relevance to the equation (0.1.3), we have the natural growth rate is ε, minimal mortality γ and as it will be shown that the bifurcation (or stabilization parameters) are taken k and β. So that, the estimate of the charcteristc time of the prey-predator system given by (0.1.3) is $\tau_c = M_i(\varepsilon^{-1}, \gamma^{-1}, (k\beta)^{-1})$ by bearing in mind that dimensions of $k\beta$, γ and ε are the same. Thus the time delay τ must satisfy $\tau < \tau_c$. This estimate can by made clear by considering the logistic equation for a biological cell population, with time delay τ. If the growth rate is ε, then we should have $\tau < \varepsilon^{-1}$. Indeed this means that a cell can not support a memory higher than its characteristic life time.

0.1.2 q-Dynamic equations modelling

To have been able to reveal different aspects of complexity in real systems, new dynamic equation models have to be introduced. New mathematical model-equations are presented in this thesis, which also take into consideration of different memory structures in a system, rather than translational memory transport. We mention that translational time delay (or translational memory transport) is considered as determinist memory. Another type of deterministic memory is the proportional memory (memory lag). The prey-predator model equatons proposed in this thesis is based on analyzing undeterminstic memory effects on the bifurcation and stability of equilibrium states. The notions of deterministic and undeterminsitic memory will be identified by defining the memory operator. By this operator, we can distinguish between the different types of memory in a dynamical system.

Due to the fact that not all environmental conditions are not of deterministic character, one may model these conditions by indeterministic time delay. Here, the q-difference operator (or q-derivative) which had been introduction by Jackson [7] is considered to measure the growth (or the decay) of the natural observable quantities that specify the dynamical system. Thus, by q-dynamic equations we mean that, the usual derivative is replaced by the q-difference (derivative). To this end, we present for q-dynamic equations that account for new memory aspects rather than memory transport or proportional memory.

The q-prey-predator model equation with self diffusion reads [8]

$$\begin{aligned}
D_q u_1(t) &= \varepsilon u_1(t)(1 - \tfrac{u_1(t)}{k}) - \tfrac{\beta u_1(t) v_1(t)}{\beta + u_1(t)} + d_1(\rho(u_2(t)) - \rho(u_1(t))),\\
D_q v_1(t) &= -\tfrac{v_1(t)(\gamma + \delta v_1(t))}{1 + v_1} + \tfrac{\beta u_1(t) v_1(t)}{\beta + u_1(t)} + d_2(\rho(v_2(t)) - \rho(v_1(t))),\\
D_q u_2(t) &= \varepsilon u_2(t)(1 - \tfrac{u_2(t)}{k}) - \tfrac{\beta u_2(t) v_2(t)}{\beta + u_2(t)} + d_1(\rho(u_1(t)) - \rho(u_2(t))),\\
D_q v_2(t) &= -\tfrac{v_2(t)(\gamma + \delta v_2(t))}{1 + v_2(t)} + \tfrac{\beta u_2(t) v_2(t)}{\beta + u_2(t)} + d_2(\rho(v_1(t)) - \rho(v_2(t))),
\end{aligned} \tag{0.1.4}$$

where

$$D_q u(t) = \begin{cases} \frac{u(t) - u(qt)}{(1-q)t}, & 0 < q < 1,\\ \frac{du(t)}{dt}, & q = 1. \end{cases}$$

It is worth noticing that the equation (0.1.4) measures the undeterministic memory effects in the rate of variation of the population density. As a new phenomena associated with q-dynamic equations is that the behavior of solutions changes abruptly to by oscillatory. This new phenomena arises from the presence of the q-constant function. Which is in contrast to “classical” equation modeling in ordinary derivative. However, if a dynamical system is stable in the “classical modeling”, it is asymptotically stable in the q-dynamic equation modeling [9].

A $q*$-(a dual to the q) dynamic equation modeling is also introduced in this thesis. The reason for studying these equations is that solutions of $q*$-equations are homomorphic to those of the q-equations. In the two cases pattern formation do occur due to the indeterminstic memory character in a dynamical system.

Returning to the equation (0.1.4), it is too complicated to determine definitely the condition for Turing instability to occur. This may be due to the fact that no characteristic, polynomial (as in the classical case) or transcendental (in the case with translational time delay), equation can be constructed. It remains to pose the criterion of relevance of the q-dynamic equation modeling in (0.1.4). We require that $\tau_q = (1 - q) < \tau_c$, where τ_c is defined in the above. We remark that when $q = 1$, then $\tau_q = 0$, and q-dynamic equations reduce to the ordinary equations.

0.1.3 Fractional dynamic equation modelling

Very recently, fractional dynamic equation modeling had been studied in the literature [10,13]. Attention was also focused on fractional reaction-diffusion equation, Burger equation and fraction Korteweg de varies equation[11-12]. In reaction diffusion systems [13], it had been pointed out that new phenomena arise, namely anomalous diffusion and anomalous transport. However, it had not been pointed out in the literature, that if the fractional modeling is relevant vis-à-vis with respect to the dynamical system under consideration. That is if the system supports a history or not. This is an interesting question to be posed. Here, this idea is discussed in order to suggest a criteria that a fractional equation modeling of a system holds or not.

Now, we consider the fractional -prey-predator equations with self diffusion between two capita , namely

$$\begin{aligned}
D^{\alpha}u_1(t) &= \varepsilon u_1(t)(1-\tfrac{u_1(t)}{k}) - \tfrac{\beta u_1(t)v_1(t)}{\beta+u_1(t)} + d_1(u_2(t)-u_1(t)),\\
D^{\alpha}v_1(t) &= -\tfrac{v_1(t)(\gamma+\delta v_1(t))}{1+v_1} + \tfrac{\beta u_1(t)v_1(t)}{\beta+u_1(t)} + d_2(v_2(t)-v_1(t)),\\
D^{\alpha}u_2(t) &= \varepsilon u_2(t)(1-\tfrac{u_2(t)}{k}) - \tfrac{\beta u_2(t)v_2(t)}{\beta+u_2(t)} + d_1(u_1(t)-u_2(t)),\\
D^{\alpha}v_2(t) &= -\tfrac{v_2(t)(\gamma+\delta v_2(t))}{1+v_2(t)} + \tfrac{\beta u_2(t)v_2(t)}{\beta+u_2(t)} + d_2(v_1(t)-v_2(t))
\end{aligned} \tag{0.1.5}$$

where the fractional derivative is taken by using the Caputo derivative that fits with smooth physical quantities, namely

$$D^{\alpha}f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int\limits_0^t (t-t_1)^{m-\alpha-1}\, f^{(m)}(t_1)\, dt_1, & m-1 \le \alpha < m, \\ f^{(m-1)}(t), & ,\alpha = m-1 \end{cases} \tag{0.1.6}$$

where $f^{(m)}$ is assumed to belong to the space $C_{\alpha}(R^{+})$ (see [14-15]).

Indeed when replacing the ordinary derivative by the fractional one's, this retains impact new effect. Indeed, the time fractional derivative of a function, by its definition in (0.1.6), measures some " memory effects" (distributed memory transport) in the growth (or decay) rate of the function. Alternatively, the fractional derivative retains the recent history of the rate of variation of the system. So that modeling dynamical systems reflects that fact. This can be seen from the kernel in the equation(0.1.6), namely $(t-t_1)^{-\alpha+m-1} = e^{(-\alpha+m-1)\ln(t-t_1)}$. Thus the characteristic time for accounting for the recent history in the growth rate is $\tau_{fr} = e^{\frac{1}{-\alpha+m-1}}$. We mention that when $\alpha = m-1$ we find that $\tau_{fr} = 0$ which is identical with the case of ordinary derivative when ($m = 2$ in (6)) . Our aim now is to justify the relevance of fractional equation modeling.

One may argue to that; the proof of existence of solution is sufficient to depict the parameters range for the rigor of fractional equation modeling. But this is not evident notably in this case, and in the cases of complex systems. In these cases intuition is invoked. We think that the characteristic time τ_{fr} has to be compared with that one of the dynamical system τ_c. For a relevant fractional equation modeling, the condition $\tau_{fr} < \tau_c$ has to hold. Indeed the characteristic time of the system τ_c depends basically on the parameters in the modeling equations. It can be inspected by analyzing the rule of the different parameters in the equations. We assume that ε is the natural growth rate, γ is the minimal mortality (cf (0.1.6)) and k and β are the dominant parameters (say the bifurcation or stabilization parameters), then $\tau_c = Min(\varepsilon^{-1}, (\beta k)^{-1}, \gamma^{-1})$. But If the system is of dimension n and the parameter space is m ($m > n$), (in (0.1.5) $n = 4 \quad , \quad m = 6$), then $\tau_c = Min(\varepsilon_i^{-1}, k_i^{-1})$ where ε_i and k_i are the growth rates and dominant parameters respectively. The former condition puts in evidence the rigor of using fractional equation modeling. If it holds that $\tau_{fr} > \tau_c$ then fractional equation modeling is irrelevant.

In contrast to the above situation, fractional (space derivative) equation modeling, in reaction diffusion systems, mechanical or physical systems, has any where a specific interpretation. For examples, when the order of the space-derivative is α and $m-1 \leq \alpha < m$, then space-fractional-modeling describes a system that changes its behavior towards to be with advection or from being with advection to be diffusive or from being diffusive to be dispersive or from being dispersive to be ultra-diffusive when $m = 1, 2, 3$ and 4 respectively. Thus, in this case space-fractional system-modeling shows the transition of the system from one state to another one‘s. Further, it may stand to explain a strange behavior of the system.