On The Performance of Some Numerical Methods for Elliptic Systems

Thesis by

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Abstract

Thesis Title: **On The Performance of Some Numerical Methods for Elliptic Systems**

The thesis focuses on the study of the systems of elliptic equations in the plane starting with the standard model of the Poisson equation and discusses two methods of generalization. In the first method, the Poisson equation of fractional order is studied. In the second method, the elliptic system is studied in general with a focus on the system consisting of two equations. The thesis is also subjected to the biharmonic equation of order four, which is transformed into elliptic system of two second order partial differential equations. Also a system of two coupled Magneto hydrodynamics, MHD, equations has been transformed to uncoupled equations, which is produced when studying the stable flow of fluid through the pipes with optional conductivity walls in the presence of a magnetic field perpendicular to the direction of flow. The thesis is subjected to two methods of numerical solution in the first method; the finite difference method is used with emphasis on implicit alternating direction method. In the second method, the weighted residual method is used using Haar functions.

Since the methods used lead to the solution of a system of linear equations, the properties of the resulting linear equations as well as some methods of obtaining these equations have been studied efficiently. It is also shown that the resulting matrix in the case of fractional order Poisson equation is close to Poisson equation in the standard form, as the fractional differential is closer to the standard image.

Key words: Poisson's Equation, Fractional Poisson's Equation, Finite Difference, Wavelet, Haar Wavelet, Alternating directional.

Summary

Partial differential equations of the most mathematical models are akin to reality and representation of natural phenomena. As well as obtaining standard solutions for equations that represent real-world models represents the particular difficulty when dealing with partial differential equations. It should be noted that there is no method of solution that can be treated with all equations, so the problem is divided into partial problems starting with the assignment and classification of auxiliary conditions and then classification of the equations. Thus, an appropriate solution can be explored. With the development of methods of approximation and instruments of scientific calculations numerical methods to solve partial differential equations have taken their place to the point that the method of finite differences, which is one of the oldest numerical methods has become one of the most used methods. In this thesis, the classification of equations and systems of partial differential equations at the plane and focus on the elliptic type is studied. The Poisson equation is the standard model for the study of numerical methods for solving partial differential equations.

In this thesis, focus is given to two methods of generalization. In the first one, the fractional order Poisson equation is studied and in the second one the system in general is studied. Attention is paid to the system consists of two equations. Moreover, we are subjected to the standard of the fourth order, which is transformed into a system of second order partial differential equations.

The thesis consists of four chapters:

Chapter One

In this chapter the definitions and basic concepts of solving elliptic partial differential equations are introduced. The study of the method of classification of second order partial differential equations in the plane is discussed. The elliptic type, which is the essence of the thesis, has been focused on. Emphasis was placed on the definition of Caputo of the fractional order differential equations. Examples of fractional order differential equations of some primary functions, which are the mainstay of the definition of solution of differential equation, have been studied. Using the finite difference method to express the fractional order derivatives, the resulting matrices and explained their relation to the standard state are studied, when the order of differentiation is an integer 2.The concept of Kronecker product is used to express matrices that represent the finite difference method in two dimensions using one dimension equation. As a result of the structural construction of the systems of the resulting equations, the iterative methods for solving the resulting

algebraic systems are studied. In this study, attention is paid to the SOR method and some of its derivatives. The concept of the use of the method of weighted residual in the existence of functions that represent the basis of the spaces of solutions to be an input to the use of wavelet method is discussed.

Chapter two:

This chapter is deduced to study the alternating direction method of solving Poisson equation of fractional order. The alternating direction implicit method is a unique way to overcome the size of calculations in multi-dimensions problems because it deals with the dimensional problems as a set of one dimension problem. This reduced the size of the system of equations, the storage space and the number of processes used. As is known, the alternating direction method is faster than the explicit method as well as the implicit method. A numerical example is given as an application to show the advantage of the method through the results of the solution. It has been shown that the results were close to the published results of the standard problem as the level of differentiation approached to the standard level.

Chapter three:

The study of how to build Haar functions and how to deal with them to solve partial differential equations with a focus on the use of wavelet method to solve the equation of Poisson in its standard image and also in the form of fractional order.
The focus of the chapter is on the use of wavelet method to solve Poisson equation in its standard form and also in fractional order. Since the system of equations resulting from the use of the method of finite differences have mathematical properties and algebraic structures characteristic, in this section the corresponding matrices when using the wavelet method are formed. And it is proved that using wavelet method results in systems of algebraic equations with matrix similar to that of the method of finite differences.
It is also shown that the resulting matrix in the case of fractional order Poisson equation is close to Poisson equation in the standard form, as the fractional differential is closer to the standard image.

Chapter four:

In this chapter, two different kinds of elliptic systems of partial differential equation are solved. The first one is obtained from transforming the biharmonic equation which of fourth order into elliptic system of two second order partial differential equations. The second one is a system of two coupled Magneto hydrodynamics, MHD, equations has been transformed to uncoupled equations. In both kinds the equations have been solved using Haar wavelet method.

All calculations were implemented using Mathematica 11.0.

Contents

Chapter One

Theory and Basic Concepts

1.1 Introduction

Solving a partial differential equation (PDE) is an important problem in general. The basic approach in the treatment of large problems in general is to divide the problem into subproblems. There is no master method that can be used to solve any PDE. So classification of partial differential equations or systems is the first step in solving a PDE problem. Classification of equations and systems of equations with canonical form representation is considered, [14], [22]. The second main subproblem is the auxiliary boundary conditions necessary to establish a well posed problem, a brief discussion about the auxiliary boundary conditions is considered, [18]. As an extension to the classical integer order derivatives appears in elliptic problems, the main topics in fractional calculus with some illustrative examples is illustrated in section 1.4, [10], [19], [31], [32]. The finite difference approximations of the derivatives of fractional orders with applications to Poisson's equation is introduced, [38], [49]. The kronecker product is used to generate the matrix representation of the two dimensional equations from the corresponding one dimensional equations, [42], in fractional order as well as in the integer case. A brief introduction to the weighted residual methods is considered. In the last section linear systems of algebraic equations and the standard stationary iterative techniques are considered, [37], [43].

1.2 Classification of Differential Equations

The subject of classification in differential equations is the compass for the researcher in this field. Knowing the type of the differential equation determines the appropriate method of solution, the simple (canonical) form, the required auxiliary conditions and moreover the characterization of the solution in advance. There are many methods in classifying differential equations; the main topic is the classification of linear second order partial differential equation or systems in the plane.

1.2.1 Classification of Second Order PDEs

It is generally accepted that the general linear second order PDE in the plane

$$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} + E(x,y)u_x + F(x,y)u_y + H(x,y)u = g(x,y), \tag{1.1}$$

is classified according to its discriminant $\Delta = B^2 - 4AC$ as hyperbolic *when* $(\Delta > 0)$ or *parabolic* when ($\Delta= 0$) or elliptic when $(\Delta < 0)$.

Our concern is the elliptic category, the representative of this category is Poisson's equation (Laplace equation). Poisson's equation appear in many applications:

In electrostatic problems the Poisson's equation is $\nabla^2 V(x, y, z) = \frac{1}{4\pi\epsilon_0}\rho(x, y, z)$, where V is the electric potential in a region with charge distribution $\rho(x, y, z)$, [21]. In Heat conduction problems the Poisson's equation, $\alpha\, \nabla^2 T(x, y) = h(x, y)$, appears as the steady state, where $T(x, y)$ represents the temperature, $h(x, y)$ represents the external heat source and α stands for the thermal conductivity. In wave propagation Poisson's equation, $\nabla^2 u(x, y) = -\frac{P(x,y)}{T}$, appears as the steady state displacement of a membrane, where $P(x, y)$ represents pressure, T represents Tension and $u(x, y)$ represents the displacement of the membrane. In Gravitational potential Poisson's equation is $\nabla^2\psi = 4\pi G\rho$, where ψ stands for Gravitational potential, G stands for universal gravitational constant and ρ stands for mass density.

Methods of Solution

There are many methods mentioned in the literature for solving partial differential equations (see [2], [21], [37]): the separation of variables, the conformal mapping, the Green's function technique and the different forms of integral transform techniques…et al.

Properties of Poisson's equation

Poisson's equation and, consequently, Laplace equation have many interesting properties that help in finding their solutions, [21]:

1. Harmonic functions: the solutions have continuous second order partial derivatives.
2. Mean value property: the value of the harmonic function at any interior point is equal to the mean value of the function on any circle in the domain with its center at that point.
3. Maximum modules property: solution of the Laplace's equation cannot have maximum or minimum in the interior of the domain. Consequently, the maximum and the minimum are taken on the boundary of the domain.
4. Analytic functions: the real and imaginary part of analytic functions are harmonic. They remain harmonic under conformal mapping so that conformal mapping becomes a powerful tool in solving boundary value problems for Laplace's equation by mapping complex domains on to simpler domains like circle.

5. Uniqueness: we can derive from the above properties that Laplace and Poisson's equation has unique stable solution for Drichilet or Neumann boundary conditions for closed surfaces, [24].

1.2.2 Classification of Homogeneous Second Order System of PDEs

The simple generalization of equation (1.1) to second order systems is the system of two equations discussed in [22]

$$A\frac{\partial^2}{\partial x^2}\binom{u}{v} + 2B\frac{\partial^2}{\partial x\partial y}\binom{u}{v} + C\frac{\partial^2}{\partial y^2}\binom{u}{v} = 0, \qquad (1.2)$$

where u and v are real functions of x, y and A, B, C are 2×2 real constant matrices.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\ B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \text{ and } C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}.$$

System (1.2) contains twelve parameters which make the system very complicated, while the canonical form for this system contains only two parameters. The system (1.2) is classified into four types namely, the elliptic, the hyperbolic, the parabolic, and the composite, [22], according to the nature of the roots of the biquadratic characteristic form.

Definition 1.1

The determinant

$$F(\xi,\eta) = |A\xi^2 + 2B\xi\eta + C\eta^2|, \qquad (1.3)$$

$$F(\xi,\eta) = (\xi^2 a_{11} + 2\xi\eta\, b_{11} + \eta^2 c_{11})(\xi^2 a_{22} + 2\xi\eta\, b_{22} + \eta^2 c_{22})$$
$$-(\xi^2 a_{12} + 2\xi\eta\, b_{12} + \eta^2 c_{12})(\xi^2 a_{21} + 2\xi\eta\, b_{21} + \eta^2 c_{21}),$$

is called the biquadratic characteristic form of the system (1.2).

The biquadratic form $F(\xi,\eta)$ has nine different combinations of roots, namely

1. Two distinct pairs of complex roots,
2. A pair of double complex roots,
3. A pair of complex roots and a double real root,
4. A pair of complex roots and two distinct real roots,
5. Four distinct real roots,
6. Three distinct real roots with one repeated ,
7. Two double real roots,

8. A triple real root and a simple real root,
9. A quadruple real root.

Elliptic systems appear in cases 1 and 2 only.

The biquadratic characteristic form may be reduced, [22], into one of the following two canonical forms:

Form (A) $\quad F(\xi,\eta) = (\xi^2 + \varepsilon\,\eta^2)(\xi^2 + k^2\eta^2)$ (1.4)

where

$$\varepsilon = 1 \; and \begin{cases} 0 < k < 1, & F(\xi,\eta) \; has \; two \; distinct \; pair \; of \; complex \; roots. \\ k = 1, & F(\xi,\eta) \; has \; pair \; of \; double \; complex \; roots. \\ k = 0, & F(\xi,\eta) \; has \; pair \; of \; complex \; roots \; and \; a \; double \; real \; roots. \end{cases}$$

$\varepsilon = 0 \; and \; k = 0, \qquad F(\xi,\eta) \; has \; a \; quadruple \; real \; root.$

Form (B) $\quad F(\xi,\eta) = \xi\eta(\delta\xi^2 + 2\alpha\xi\eta + \varepsilon\eta^2),$

where

$$\delta = \varepsilon = 1 \text{ and } \begin{cases} 0 \le \alpha < 1, & F(\xi,\eta) \; has \; a \; pair \; of \; complex \; and \\ & two \; distinct \; real \; roots. \\ \alpha = 1, & F(\xi,\eta) \; has \; three \; distinct \; real \; roots. \\ \alpha > 1, & F(\xi,\eta) \; has \; four \; distinct \; real \; roots. \end{cases}$$

$\delta = \varepsilon = 0 \; and \; \alpha = 1, \qquad F(\xi,\eta) \; has \; two \; double \; real \; roots.$

$\delta = 1 \; and \; \varepsilon = \alpha = 0, \qquad F(\xi,\eta) \; has \; a \; triple \; real \; roots.$

Corresponding to the biquadratic characteristic form (A), Eq. (1.4), the canonical form of system (1.2) is

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\partial^2}{\partial x^2} + 2\begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}\frac{\partial^2}{\partial x \partial y} + \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\frac{\partial^2}{\partial y^2}\right]\begin{pmatrix} u \\ v \end{pmatrix} = 0, \qquad (1.5)$$

$\lambda + \mu - 4b = k^2 + \varepsilon;\; \lambda\mu = k^2\varepsilon, b \neq 0,$ where $\varepsilon = 0{,}1 \; and \; 0 \le k \le 1.$

(1) If $\lambda \neq \mu$

(a) when $\varepsilon = 1 \; and \; 0 < k < 1,$ then $F(\xi,\eta) = 0$ has two distinct pairs of complex roots.

(b) when $\varepsilon = 1 \; and \; k = 1,$ then $F(\xi,\eta) = 0$ has a pairs of double complex roots.

(c) when $\varepsilon = 1,\ k = 0$, then $F(\xi,\eta) = 0$ has a pair of complex roots and a real double root.

(d) When $\varepsilon = 0,\ k = 0$, then $F(\xi,\eta) = 0$ has a real quadruple root.

(2) If $\lambda = \mu$, then the only possibility is that $\varepsilon = 1,\ 0 \leq k \leq 1$ and $b < 0$. This is also holds in cases (1), (2) and (3) for $\lambda \neq \mu$.

Here the study (elliptic systems) will focus only on the canonical form (A).

1.2.3 General Elliptic Systems of PDEs

In [14] it is mentioned that Douglis and Nirenberg introduced a general definition for elliptic systems. The definition depends on writing the linear differential operator in matrix representation and the ellipticity of the system depends on the value of the associated determinant. Let Ω be a bounded domain in R^d and consider a system of n PDEs, written in the matrix form

$$L_\Omega = F_\Omega, \tag{1.6}$$

where L_Ω is the matrix of differential operators:

$$L_\Omega = \begin{bmatrix} L_{11} & \cdots & L_{1n} \\ \vdots & \ddots & \vdots \\ L_{n1} & \cdots & L_{nn} \end{bmatrix},$$

where

$L_{ij} = L_{ij}(D) = \sum_{|\alpha| \leq k} c_\alpha D^\alpha$,

$(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d), D^\alpha = (\partial/\partial x_1)^{\alpha_1} (\partial/\partial x_2)^{\alpha_2} \ldots\ (\partial/\partial x_d)^{\alpha_d}\ )$

$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$,

can be considered as a polynomial in $D = \frac{\partial}{\partial\, x}$. L_{ij} is of order k, if there exists $c_\alpha \neq 0$ with $|\alpha| = k$. Assume that there exist two n indices m_i and m'_j ; $1 \leq i, j \leq n$ with $2m = \sum_{i,j=1}^{n} (m_i + m'_j)$ such that

$$\text{order of } L_{ij} \leq m_i + m'_j \tag{1.7}$$

For given m_i and m'_j , define the principal term $L^P_{ij}(D) = \sum_{|\alpha| = m_i + m'_j} c_\alpha D^\alpha$.

Note that $L^P_{ij}(D)$ may vanish although $L_{ij} \neq 0$. The matrix with entries $L^P_{ij}(D)$ yields the principal part $L^P_\Omega(D)$ of $L_\Omega(D)$. Replacing D by $\xi = (\xi_1, \ldots, \xi_d)$ one obtain a matrix -valued polynomial $L^P(\xi)$. If the coefficients of L^P_Ω depend on $x \epsilon R^d$, write $L^P(x,\xi)$ instead of $L^P(\xi)$. The ellipticity is defined by means of the determinant

$$\lambda(x,\xi) = \det L^P(x,\ \xi).$$

The definition of elliptic systems of PDEs is given as follows