

Cairo University

Faculty of Economics and Political Science

Department of Statistics

Estimation of the Burr-XII Distribution for Partially Accelerated Life Tests Using Censored Data

By

Dalia Ahmed Omar

Supervised by

Prof. Abdalla A. Abdel-Ghaly

Professor of Statistics

Dr. Zeinab H. Amin

Associate Professor of
Statistics

Department of Statistics

Faculty of Economics and Political Science,

Cairo University

A thesis submitted to the department of statistics to fulfill the requirements of the Master of Science degree in statistics

2009

Cairo University

Faculty of Economics and Political Science

Department of Statistics

Cairo University

Faculty of Economics and Political Science

Department of Statistics

Name: *Dalia Ahmed Mohamed Omar*

Nationality: *Egyptian*

Date of birth: *1/ 6 / 1979*

Specialization: *Reliability and Life Testing*

Supervisors: ***Prof. Abdalla A. Abdel-Ghaly***, Professor of Statistics, Department of Statistics,
Faculty of Economics and Political Science, Cairo University.

Dr. Zeinab H. Amin, Associate Professor of Statistics, Department of Statistics,
Faculty of Economics and Political Science, Cairo University.

Title: *Estimation of the Burr-XII Distribution for Partially Accelerated Life Tests Using
Censored Data*

Summary

Running test units under normal use conditions to obtain life data is sometimes impractical and difficult especially with the long lifetime of today's products. Given this difficulty, accelerated life testing (ALT) has been used to enable the practitioner to test the life time of the products by subjecting these products to severe levels of stress that are not attainable at normal use conditions in order to induce early failures.

The main assumption in ALT is that the model describing the relationship between life and stress is known. In some cases, the nature of this life-stress relationship is unknown; that is, ALT data cannot be extrapolated to normal use conditions. In such cases, partially accelerated life testing (PALT) is the most reasonable scheme to use in estimating the acceleration factor and thus extrapolating the accelerated data to normal use conditions.

The thesis considered the estimation problem using the maximum likelihood and Bayesian approaches for the parameters of the Burr-XII distribution as well as the acceleration factor under step-stress PALT and constant-stress PALT using types-I and II censoring. The Bayesian approach is applied assuming a non-informative prior (Jeffreys' rule) and the informative priors for the two parameters of our lifetime model.

Following the introductory chapter, the thesis is divided into two main parts. Part one of the thesis deals with Step-Stress PALT. Chapter II considers the maximum likelihood method in estimating the parameters of the Burr-XII distribution and the acceleration factor under both type-I and type-II censoring. The performance of the estimates has been measured using the relative absolute bias, the mean square error and the relative error of the maximum likelihood estimates. Also the asymptotic variances and covariances are obtained. Chapter III presents the Bayesian approach to estimate the unknown parameters in the case of type-I and type-II censored samples. The Bayes estimates and their posterior variances are obtained in the case of non-informative prior while the case of informative priors is discussed theoretically. Also, the relative absolute bias, the mean square error and the relative error of the Bayes estimates are evaluated.

Part two of the thesis focus on the second scheme of PALT which is the constant-stress PALT. Chapter IV discusses the evaluation of the maximum likelihood estimates for the parameters of our lifetime model and the acceleration factor under type-I and type-II censoring. The asymptotic variances and covariances of the maximum likelihood estimates are obtained. Chapter V deals with the problem of estimation using the Bayesian approach using type-I and type-II censored data where the Bayes estimates and their posterior variances are obtained. The relative absolute bias, the mean square error and the relative error of the maximum likelihood and Bayes estimates are also evaluated. Finally, chapter VI is devoted to summarizing conclusions and points for future work.

Abstract

This thesis deals with the problem of estimation concerning the parameters of the Burr type XII distribution. The unknown parameters are estimated under both step and constant stress PALT for types I and II censored samples. The first part of the thesis is devoted to the step-stress PALT. Maximum likelihood estimators and their asymptotic variances are developed for the parameters of the distribution as well as for the acceleration factor. Also the Bayesian approach is used to estimate the unknown parameters of our life time distribution and the acceleration factor in the case of type-I and type-II censored samples. The Bayes estimates and their posterior variances are obtained in the case of non-informative prior while the case of informative priors is discussed theoretically.

Part two of the thesis focus on the second scheme of PALT which is the constant-stress PALT. The maximum likelihood estimators and the asymptotic variances of the parameters of our model and the acceleration factor are obtained under type-I and type-II censoring. Estimation using the Bayesian approach using type-I and type-II censored data is considered. The posterior variances of the Bayes estimates are obtained to evaluate their performance in the case of non-informative prior. The case of informative priors is discussed theoretically.

At the end of each chapter, a simulation study is presented to investigate the efficiency of the estimates and to illustrate the theoretical results under the different sampling schemes for various sample sizes.

Key words: Step stress; Constant stress; Partially accelerated life tests; Type I censoring; Type II censoring; Burr type XII distribution; Maximum likelihood estimation; Bayesian approach; Non Informative prior; Informative priors; Newton-Raphson Algorithm.

Acknowledgment

I would like to express my greatest and deepest gratitude to Prof. **Abdalla A. Abdel-Ghaly**, Department of Statistics, Faculty of Economics and Political Science, Cairo University, for his great support, valuable guidance, fruitful advice and continuous encouragement during the preparation of this thesis. His parental kindness helped me to accomplish this study.

My profound appreciation and gratitude goes to Dr. **Zeinab H. Amin**, Department of Statistics, Faculty of Economics and Political Science, Cairo University, for her great help, kind supervision, valuable comments, constructive criticism and creative suggestions. Her efforts will never be forgotten.

CONTENTS

	Page
Acknowledgement	iv
Abstract	v
 CHAPTER I: INTRODUCTION	 1
1.1 Partially Accelerated Life Testing	2
1.2 Literature Review on Partially Accelerated Life Testing	4
1.3 The Burr-XII Distribution as a Life Time Model	5
1.4 Sampling Theory versus Bayesian Approach in Statistical Inference	7
1.4.1 Sampling Theory (Maximum Likelihood Method)	7
1.4.2 Bayesian Approach	8
1.5 Outline of the Thesis	10
PART ONE STEP-STRESS PARTIALLY ACCELERATED LIFE TETING	
 CHAPTER II: MAXIMUM LIKELIHOOD ESTIMATION UNDER STEP-STRESS PARTIALLY ACCELERATED LIFE TESTING	 12
2.1 Introduction	12
2.2 Maximum Likelihood Estimation under Type-I Censoring	12
2.2.1 The Maximum Likelihood Estimates	12
2.2.2 Illustrative Example	17
2.2.3 Simulation Study	18
2.3 Maximum Likelihood Estimation under Type-II Censoring	22
2.3.1 The Maximum Likelihood Estimates	22
2.3.2 Illustrative Example	24
2.3.3 Simulation Study	26

CHAPTER III: BAYESIAN ESTIMATION UNDER STEP- STRESS PARTIALLY ACCELERATED LIFE TESTING 29

3.1	Introduction	29
3.2	Bayesian Estimation under Type-I Censoring	30
3.2.1	Non-informative Prior	31
3.2.2	Informative Priors	35
3.2.2.1	Continuous Conjugate Prior for the Shape Parameters	35
3.2.2.2	Discrete Prior for the Shape Parameter c	37
3.2.3	Simulation Study	39
3.3	Bayesian Estimation under Type-II Censoring	43
3.3.1	Non-informative Prior	43
3.3.2	Informative Priors	45
3.3.2.1	Continuous Conjugate Prior for the Shape Parameters	45
3.3.2.2	Discrete Prior for the Shape Parameter c	46
3.3.3	Simulation Study	48

PART TWO CONSTANT-STRESS PARTIALLY ACCELERATED LIFE TESTING

CHAPTER IV: MAXIMUM LIKELIHOOD ESTIMATION UNDER CONSTANT-STRESS PARTIALLY ACCELERATE LIFE TESTING 52

4.1	Introduction	52
4.2	Maximum Likelihood Estimation under Type-I Censoring	53
4.2.1	The Maximum Likelihood Estimates	53
4.2.2	Illustrative Example	55
4.2.3	Simulation Study	57
4.3	Maximum Likelihood Estimation under Type-II Censoring	60
4.3.1	The Maximum Likelihood Estimates	60
4.3.2	Illustrative Example	62
4.3.3	Simulation Study	63

CHAPTER V:	BAYESIAN ESTIMATION UNDER CONSTANT-STRESS PARTIALLY ACCELERATED LIFE TESTING	66
5.1	Introduction	66
5.2	Bayesian Estimation under Type-I Censoring	66
5.2.1	Non-informative Prior	67
5.2.2	Informative Priors	70
5.2.2.1	Continuous Conjugate Prior for the Shape Parameters	70
5.2.2.2	Discrete Prior for the Shape Parameter c	72
5.2.3	Simulation Study	74
5.3	Bayesian Estimation under Type-II Censoring	77
5.3.1	Non-informative Prior	77
5.3.2	Informative Priors	80
5.3.2.1	Continuous Conjugate Prior for the Shape Parameters	80
5.3.2.2	Discrete Prior for the Shape Parameter c	81
5.3.3	Simulation Study	83
CHAPTER VI:	SUMMARY and CONCLUSION	86
6.1	Summary	86
6.2	Points for Future Work	87
REFERENCES		89

Chapter I

Introduction

Traditional life data analysis involves analyzing failure data obtained under normal operating conditions in order to quantify the life characteristics of the product. However, in many situations, the long life time of today's products makes it extremely difficult for the practitioner to obtain such life data. Given this difficulty, accelerated life testing (ALT) has been used to enable the practitioner to test the life time of the products by subjecting these products to severe levels of stress that are not attainable at normal use conditions in order to induce early failures. The purpose of using ALT is to predict the life of the product at normal use conditions from data obtained in accelerated life tests.

In some tests, the test unit is subject to a single stress. In other tests a single stress is not enough to cause the failure of the unit during the appropriate time, so multiple stresses must be used to perform the test in a limited time period. The stress can be applied in several ways. The two commonly used ways are:

- **Constant Stress (time independent stress):**

In this case the stress applied on a sample of units does not vary with time and it continues until failure or censoring whichever occurs first. As indicated by Nelson (1980), this type of stress is widely used and preferred because:

- a) In most applications the stress is constant.
- b) It is much easier to apply and quantify constant stress.
- c) Models for constant stress are available, widely publicized and empirically verified.
- d) Extrapolation from constant stress test is more accurate than that from time dependent stress test.

However, sometimes testing at constant stress may take too much time because there is usually a large variation in failure times. In this case, a method that assures quicker

occurrence of the failures is needed. In this case, step stress testing appears to be more efficient than constant stress.

- **Step Stress (time dependent stress):**

Under this scheme, the stress setting of the test units is changed at pre-specified times. Generally, a test unit is first subjected to a specified low stress (usually the normal use stress). If the unit does not fail at a specified time, the level of stress to which it is exposed is raised and held for a specific time period. Stress is repeatedly increased and held until the test unit fails or the censoring condition is reached.

The main assumption in ALT is that the model describing the relationship between life and stress is known. The knowledge of the nature of this relationship allows the practitioner to estimate the lifetimes at design stress. In some cases, the nature of this life-stress relationship is unknown; that is, ALT data cannot be extrapolated to normal use conditions. In such cases, partially accelerated life testing (PALT) is the most reasonable scheme to use in estimating the acceleration factor and thus extrapolating the accelerated data to normal use conditions. The acceleration factor β ($\beta > 1$) is the ratio of the hazard rate at accelerated conditions to that at normal use conditions.

1.1 Partially Accelerated Life Testing (PALT)

In ALT, the test items are run only at accelerated conditions. PALT combines both ordinary and accelerated life testing procedures, that is, the test units are run at both normal use and accelerated conditions. The main assumption in PALT is that the life-stress relationship is unknown or cannot be assumed. It is also used when it is desired to test the unit at a certain accelerated condition and then extrapolate the data to normal conditions. An extra advantage of PALT is that more failure data are obtained in a limited time without subjecting all the units to high levels of stress. There are two basic ways for applying PALT:

i. Step-Stress PALT

The concept of step-stress PALT was introduced by Goel (1971). Under step-stress PALT, the test units are run at normal conditions for a pre-specified time τ (stress change

time) and if a test unit survives time τ , it is then run under accelerated conditions until it either fails or is censored. The effect of switching the normal stress to a higher level of stress is to multiply the remaining life time of the unit by the inverse of the acceleration factor β .

ii. Constant-Stress PALT

This type of PALTs was introduced by Bai and Chung (1992). Under constant-stress PALT, each test unit is either run at normal use conditions or accelerated conditions. The n sample units are divided into two groups according to a certain proportion p such that $[np]$ units are randomly selected and allocated to run under normal use conditions, while the remaining $[n(1-p)]$ units are allocated to run under accelerated conditions. The lifetime of a unit run at normal use conditions is denoted by T , and the lifetime of that unit run at accelerated conditions is denoted by $Z = \beta^{-1}T$. Each test unit is tested until failure or censoring and the test conditions are not changed. The lifetimes $T_i, i = 1, \dots, [np]$ of the test units allocated to normal use conditions and the lifetimes $Z_j, j = 1, \dots, [n(1-p)]$ of the test units allocated to accelerated conditions are independent and identically distributed. Also T_i and Z_j are mutually independent.

The analysis can be carried out using either complete data - The lifetime of each test unit in our sample is known - or censored data. Censoring occurs when the exact lifetimes are known only for a portion of the units under study.

In this study, two common types of censoring will be used in the analysis; type-I and type-II. Under type-I censoring, the PALT is applied on the whole n units for a pre-specified time η (censoring time) and when the time η is reached, the test will stop and the failure times of the units that failed are obtained. The units that survived past time η are considered censored. The number of failures here is a random variable while the test duration η is fixed. Under type-II censoring, PALT is performed on the n test units of the sample until a pre-specified number of units- say r - out of the n units fail and then the test is terminated. The test duration η is a random variable and the number of failures r is fixed.

1.2 Literature Review on Partially Accelerated Life Testing

PALT has been considered by a vast number of authors, see for example Degroot and Goel (1979) who introduced a method for life testing in which an item is tested under normal use conditions for a pre-specified time period and if the item does not fail by this time it is subjected to a higher level of stress. This increase in the stress level has a multiplicative effect on the remaining life of the test item by some factor which is the inverse of the acceleration factor. The Bayesian approach was used by Degroot and Goel (1979) to estimate the parameters of an exponential distribution and the acceleration factor using step PALT in case of complete data. They also obtained an optimal design of the PALT.

Bhattacharyya and Soejoeti (1989) proposed the Tampered failure rate (TFR) model for step-stress ALT. That model assumed that the increase of the stress has a multiplicative effect on the failure rate function over the remaining life of the test unit. The parameters of a Weibull distribution and the acceleration factor were estimated using maximum likelihood method in step-stress ALT in case of complete sampling.

Bai and Chung (1992) derived the estimates of the exponential scale parameter and the acceleration factor of both constant and step PALT using maximum likelihood method of estimation under type-I censoring. They also considered optimal designs for the two PALTs. Bai et al. (1993) used the maximum likelihood method to derive the estimates for the parameters of a lognormal distribution and the acceleration factor in a step PALT using type-I censored sample.

Attia et al. (1996) considered the problem of estimating the parameters of the Weibull distribution and the acceleration factor for step PALT using the method of maximum likelihood in case of type-I censored data. Attia et al. (1997) used the Bayesian approach in estimating the parameters of the Weibull distribution and the acceleration factor in step PALT in both type- I and type- II censoring given that the shape parameter is known.

Abdel-Ghani (1998) obtained the estimates for the Weibull distribution parameters and the acceleration factor using both classical and Bayesian approaches. Type-I and type-II censored data were used in estimation through step and constant PALT.

Attia et al. (2002) considered the estimation problem in step PALT. The maximum likelihood estimates are obtained for the Weibull distribution parameters and the acceleration factor in both type-I and type-II censored samples.