Ain Shams University
Faculty of Education
Department of Mathematics

Investigations in Differential and Integral Equations and some Applications

A Thesis

Submitted in Partial Fulfillment of the Requirements for the Master's Degree in Teacher Preparation in Science

(Pure mathematics)

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Summary

Integral equations are encountered in various fields of science and numerous applications (in elasticity, plasticity, heat and mass transfer, oscillation theory, fluid dynamics, filtration theory, electrostatics, electrodynamics, biomechanics, game theory, control, queuing theory, electrical engineering, economics, medicine, etc.).

Exact (closed-form) solutions of integral equations play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural science. Lots of equations of physics, chemistry, and biology contain functions or parameters which are obtained from experiments and hence are not strictly fixed. Therefore, it is expedient to choose the structure of these functions so that it would be easier to analyze and solve the equation. As a possible selection criterion, one may adopt the requirement that the model integral equation admits a solution in a closed form. Exact solutions can be used to verify the consistency and estimate errors of various numerical, asymptotic, and approximate methods.

Through out this thesis we come across many subjects that handle the linear integral and nonlinear integro-differential equations by using the modern mathematical methods, and some of the powerful traditional methods.

This thesis will be powerful tool for those who are concerned by applied mathematics, physical science, and engineering, attempts are made so that it presents both analytical and numerical approaches in a clear and systematic fashion to make this thesis accessible to those who work in these fields.

Also there will be a part that is devoted to thoroughly examine the nonlinear integral equations and its applications. Mathematical physics models, such as diffraction problems, scattering in quantum mechanics, conformal mapping, and water waves also contributed to the creation of nonlinear integral equations. Because it is not always possible to find exact solutions to problems of physical science that are posed, much work is devoted to obtaining qualitative approximations that highlight the structure of the solution and we worked to search for new methods to become closer or to obtain the exact solution.

Summarizing this thesis demonstrates the following five chapters

Chapter 1: This chapter is an introduction to the basic definitions and concepts of in-

tegral and integro-differential equations with some important definitions and theorems which are necessary for studying the properties of integral and integro-differential equations. It also contains an introduction to some methods used in solving integral equations.

Chapter 2: In this chapter, we introduce the basic definitions and theorems of the Differential transform method (DTM) for some linear and non-linear functions, it also contains differential transform for convolution theorem and error analysis. Also, we investigate the differential transform method for the Fresnel integrals, Singularly perturbed Volterra integral equations, Volterra population model and a system of differential equations.

Exact and approximate solutions were obtained by the mentioned method and we compared those solutions with previous work that used other methods for the same model.

Chapter 3: In this chapter, we apply the differential transform method for solving quadratic integral equations to find the exact solutions for some models and problems, also we study the existence and uniqueness theorem for quadratic integral equations. It also contains applications like singularly perturbed Volterra integral equations and Heat Radiation in a Semi-Infinite Solid and their solution using differential transform method.

On the other hand, we introduce the basic idea of Homotopy Perturbation Method (HPM) and investigate the method for Singularly perturbed Volterra integral equations, quadratic Singularly perturbed Volterra integral equations, heat radiation in a semi-infinite solid application, and Volterra population model to find exact and approximate solutions for these applications. The method showed remarkable resulted and was compared to the results of another methods used to solve the same applications.

Chapter 4: In this chapter, we study the differential transform method to find the approximate solutions of nonlinear delay differential equations (DDEs) and delay integro-differential equations of type $u(qt-\tau)$ and $u(qt)$. Also, we prove theorems related to the differential transformation of the delay functions $u(qt-\tau)$ and $u(qt)$. The results of some examples were tested by applying the DTM showed remarkable performance through a comparison with the pervious results.

Chapter 5: This chapter introduces the basic definitions and theorems of two and three-dimensional differential transform for integral equations. By applying the differential transform method, the integral equations can be transformed to an algebraic equations and solving this equations, we find the approximate solutions of the integral equations.

Also, we apply two and three-dimensional differential transform on some Integral equations and we compared the results with the exact solutions.

Two papers were published from the work in this chapter entitled:

1- "Applications on Differential Transform Method for Solving Singularly Perturbed Volterra Integral Equation, Volterra Integral Equation and Integro-differential Equation." in the International Journal of Mathematics Trends and Technology (IJMTT), (2015) [41].

2- "Applications on Differential Transform method for solving Singularly Perturbed Volterra integral equation, Volterra integral equation and integro-differential equation." in the Journal of Fractional Calculus and Applications, (2014) [42].

A third paper was accepted containing some work from these chapters entitled:

3- "Exact Solutions of Quadratic Integral and Integro-differential equations." in the Journal of Nonlinear Analysis and Optimization, (2015) [43].

Contents

Chapter 1

Preliminaries and Auxiliary Tools

1.1 Integral Equations

Integral equations [45] are encountered in various fields of science and numerous applications (in elasticity, plasticity, heat and mass transfer, oscillation theory, fluid dynamics, filtration theory, electrostatics, electrodynamics, biomechanics, game theory, control, queuing theory, electrical engineering, economics, medicine, etc.).

An integral equation is an equation in which the unknown function $u(x)$ appears under an integral sign. A standard integral equation in $u(x)$ is of the form [45]

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt, \tag{1.1}$$

where $g(x)$ and $h(x)$ are the limits of integration, λ is a constant parameter, and $K(x,t)$ is a function of two variables x and t called the kernel or the nucleus of the integral equation. The function $u(x)$ that will be determined appears inside the integral sign and outside the integral sign as well. The functions $f(x)$ and $K(x,t)$ are given in advance. It is to be noted that the limits of integration $g(x)$ and $h(x)$ may be both variables, constants, or mixed.

Integral equations [45] appear in many forms. Two distinct ways that depend on the limits of integration are used to characterize integral equations in eq.(1.1), namely:

1. If the limits of integration are fixed, the integral equation is called a **Fredholm integral**

equation and is given in the form

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, \tag{1.2}$$

where a and b are constants.

The unknown function $u(x)$ may appear only inside integral equation in the form

$$f(x) = \int_a^b K(x,t)u(t)dt. \tag{1.3}$$

2. If at least one limit is a variable, the equation is called a **Volterra integral equation** and is given in the form

$$u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt, \tag{1.4}$$

where a is constant.

The unknown function $u(x)$ may appear only inside integral equation in the form

$$f(x) = \int_0^x K(x,t)u(t)dt. \tag{1.5}$$

The two other distinct kinds, that depend on the appearance of the unknown function $u(x)$, are defined as follows

1. If the unknown function $u(x)$ appears only under the integral sign of Fredholm or Volterra equation, the integral equation is called a **first kind Fredholm** or **Volterra integral equation** respectively as in eq.(1.3) and (1.5).

2. If the unknown function $u(x)$ appears both inside and outside the integral sign of Fredholm or Volterra equation, the integral equation is called a second kind Fredholm or Volterra equation integral equation respectively as in eq.(1.2) and (1.4).

Volterra-Fredholm Integral Equations

The Volterra-Fredholm integral equations [44], [32] arise from parabolic boundary value problems, from the mathematical modelling of the spatiotemporal development of an epidemic, and from various physical and biological models. The Volterra-Fredholm integral equations appear in the literature in two forms, namely

$$u(x) = f(x) + \lambda_1 \int_a^x K_1(x-t)u(t)dt + \lambda_2 \int_a^b K_2(x,t)u(t)dt, \tag{1.6}$$

and

$$u(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F(x,t,\zeta,\tau,u(\zeta,\tau))d\zeta d\tau, (x,t) \in \Omega \times [0,T], \tag{1.7}$$

where $f(x,t)$ and $F(x,t,\zeta,\tau,u(\zeta,\tau))$ are analytic functions on $D = \Omega \times [0,T]$, and Ω is a closed subset of $\mathbb{R}^n$, $n = 1,2,3$. It is interesting to note that (1.6) contains disjoint Volterra and Fredholm integral equations, where as (1.7) contains mixed Volterra and Fredholm integral equations. Moreover, the unknown functions $u(x)$ and $u(x,t)$ appear inside and outside the integral signs. This is a characteristic feature of a second kind integral equation. If the unknown functions appear only inside the integral signs, the resulting equations are of first kind.

1.2 Integro-differential Equations

Integro-differential equations appear in many scientific applications, especially when we convert initial value problems or boundary value problems to integral equations. The integro-differential equations contain both integral and differential operators. The derivatives of the unknown functions may appear to any order. In classifying integro-differential equations, we follow the same category used before according to the limits of integration.

An integro-differential equation is an equation in which the unknown function $u(x)$ appears under an integral sign and contains an ordinary derivative $u^{(n)}(x) = \frac{d^n u(x)}{dx^n}$ as well. A standard integro-differential equation is of the form [44]

$$u^{(n)}(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt, \tag{1.8}$$

where $g(x)$, $h(x)$, $f(x)$, λ and the kernel $K(x,t)$ are as prescribed before in eq.(1.1).

1. Fredholm Integro-Differential Equations

Fredholm integro-differential equations [45] appear when we convert differential equations to integral equations. The Fredholm integro-differential equation contains the unknown function $u(x)$ and one of its derivatives $u^{(n)}(x)$, $n \geq 1$ inside and outside the integral sign respectively. The limits of integration in this case are fixed as in the Fredholm integral equations. The equation is labeled as integro-differential because it contains differential and integral operators in the same equation. It is important to note that initial conditions should be given for Fredholm integro-differential equations to obtain the particular solutions. The Fredholm integro-differential equation appears in the form

$$u^{(n)}(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, \tag{1.9}$$

other derivatives of less order may appear with $u^{(n)}$ at the left side.

2. Volterra Integro-differential Equations

Volterra integro-differential equations [45] appear when we convert initial value problems (IVP) to integral equations. The Volterra integro-differential equation contains the unknown function $u(x)$ and one of its derivatives $u^{(n)}(x)$, $n \geq 1$ inside and outside the integral sign. At least one of the limits of integration in this case is a variable as in the Volterra integral equations. It is important to note that initial conditions should be given for Volterra integro-differential equations to determine the particular solutions. The Volterra integro-differential equation appears in the form

$$u^{(n)}(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt, \tag{1.10}$$

other derivatives of less order may appear with $u^{(n)}(x)$ at the left side.

3. Volterra-Fredholm Integro-differential Equations

The Volterra-Fredholm integro-differential equations [45] appear in the literature in two forms, namely

$$u^{(n)}(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)u(t)dt + \lambda_2 \int_a^b K_2(x,t)u(t)dt, \tag{1.11}$$

and

$$u^{(n)}(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F(x,t,\zeta,\tau,u(\zeta,\tau))d\zeta d\tau, (x,t) \in \Omega x\left[0,T\right], \tag{1.12}$$

where $f(x,t)$ and $F(x,t,\zeta,\tau,u(\zeta,\tau))$ are analytic functions on $D = \Omega \times\ [0,T]$, and Ω is a closed subset of $\mathbb{R}^n$, $n = 1,2,3$.

1.3 Linearity and Homogeneity

Integral equations and integro-differential equations fall into two other types of classifications according to linearity and homogeneity concepts. These two concepts play a major role in the structure of the solutions. In what follows we highlight the definitions of these concepts.

1.Linearity concept

If the exponent of the unknown function $u(x)$ inside the integral sign is one, the integral equation or the integro-differential equation is called linear [44]. If the unknown function $u(x)$ has exponent other than one, or if the equation contains nonlinear functions of $u(x)$, such as e^u, $sinhu$, $cosu$, $ln(1+u)$, the integral equation or the integro-differential equation is called nonlinear. The nonlinear Volterra integral equation of the second kind is represented by the form

$$u(x) = f(x) + \int_0^x K(x,t)F(u(t))dt,$$

where $F(u(t))$ is nonlinear function of $u(x)$.

Also, the nonlinear volterra integro-differential equation of the second kind is represented