
**Ain Shams University**
**University College for Women**
**Mathematics Department**

# On Stochastic Integer Linear Fractional Programming Problems

*Thesis*
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*by*

**Yehia Abdel-Azeem Moustafa Aly**

***Supervisors***

**Dr. Soraya A. E. Sherif**
Professor of Pure Mathematics
University College for Women
Ain Shams University

**Dr. Ebrahim A. Youness**
Professor of Pure Mathematics
Faculty of Science
Tanta University

**Dr. Omar M. Saad**
Assistant Professor of Pure Mathematics
Faculty of Science
Helwan University

**Dr. Karima M. Rabo**
Lecturer of Pure Mathematics
University College for Women
Ain Shams University

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## ABSTRACT

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The main purpose of this thesis is to present a study on stochastic integer linear fractional programming problems.

This thesis contains six chapters.

In the first chapter, we summarized definitions, notions and known results which are needed in the thesis.

In the second chapter, we discussed integer linear fractional programming problems with random parameters in the objective function. An algorithm is proposed for treating the problem of concern.

In the third chapter, we introduced the integer linear fractional programming problems with random parameters in the constraints. Also, an algorithm is proposed for treating the problem.

In Chapter four, we are concerned with the integer linear fractional programming problems with stochastic parameters in the objective function and constraints. An algorithm is proposed for treating the problem.

In Chapter five, we introduced some basic notions of integer linear fractional programming problems with random parameters in the objective function and in the constraints. These notions are the set of feasible parameters, the solvability set and the stability set of the first kind. Algorithms are proposed and described in finite steps to determine the stability sets of the first kind for such problems.

In Chapter six, we presented the conclusions related to study in the thesis and also some suggestions for future research in the area of stochastic integer linear fractional programming problems.

**Keyword:** Integer programming, Fractional programming, Stochastic programming, Solvability set, Stability set of the first kind.

# CHAPTER 1

## Introduction and Basic Concepts

In some application areas, such as economics, management, military purposes and control theory, mathematical optimization theory plays an important role. The mathematical optimization problem in which some, but not all of the variables required to be an integer, is a mixed-integer optimization problem (MIP). If all the variables are integers the resulting problem is a pure integer optimization problem (IP). When the data needed in these problems, are not exactly known, the problems are called stochastic integer optimization problems.

A stochastic integer optimization problem was studied by Slyke and Wets [42]. They proposed the L-shaped algorithm based on a dual decomposition scheme and Lagrangian relaxation. They have converted the problem to a separable problem which is much easier to be solved. Wollmer [46] presented an algorithm for models with 0-1 first stage variables and continuous second-stage problem and Laporte and Louveaux [27] extended this algorithm to models with binary first-stage variables and arbitrary second-stage problem. Carøe and Tind [3] considered L-shaped decomposition for integer recourse problems in general setting. They allowed arbitrary first-stage variable in combination with a pure integer second-stage problem. Carøe and Schultz [4] proposed the dual decomposition algorithm which is applied to two-stage problems with mixed-integer variables in both stage.

The work of this thesis is mainly devoted on studying methods of so-

lution of stochastic integer fractional programming problems. A parametric study is also given. To deal with this kind of problems, let us summarize some basic definitions and results which will be needed later throughout this thesis.

## 1.1 Random Variables and Measurable Sets

In this section, we summarize some basic definitions and results on random variables and measurable spaces

**Definition 1.1.1 [12].** In a random experiment, a number $p(E)$ which indicates the probability that an event $E$ in the sample space $S$ will be occur must satisfy the following axioms:

(i) $p(E_i) \geq 0, \ \forall E_i \in S$

(ii) $p(S) = 1$

(iii) $p(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} p(E_i)$ for any infinite sequence of disjoint events $E_1, E_2, \ldots$

**Definition 1.1.2 [12].** Let $X$ be a random variable. A function $f(x)$ that assigns for each value $X$ of $X$ a number $p(x) \in [0, 1]$ is called probability density function of $X$.

A random variable $X$ is called discrete if it has discrete values and is called continuous if its values are in continuous interval.

**Definition 1.1.3 [12].** Let $X$ be a random variable, characterized by a function $u(x)$, has density function $f(x)$ . An expectation of $X$ is denoted by $E(X)$ and is defined as:

$$E(X) = \sum_{x} u(x) f(x) \text{ (if } X \text{ is discrete)}$$

$$E(X) = \int_{-\infty}^{\infty} u(x)f(x)dx \text{ (if } X \text{ is continuous).}$$

The expectation of $X$ is also called the mean of $X$ which is denoted by $\mu$, i.e., $E(X) = \mu$.

**Definition 1.1.4 [12].** Suppose that $X$ is a random variable with mean $\mu$ .The variance of $X$, is denoted by $var(X)$ or $\sigma^2(x)$, and is defined by

$$\sigma^2(x) = var(x) = E[(X - \mu)^2] = E(x^2) - \mu^2.$$

**Definition 1.1.5 [12].** Let $X, Y$ be two random variables that have joint probability density function $f(x, y)$ and let $E(x)$ and $E(y)$ are the mean of $X$ and the mean of $Y$ respectively, so the covariance of $X$ and $Y$, is denoted by $cov(X, Y)$ , and is defined, if it exists, by :

$$cov(X, Y) = E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y).$$

**Definition 1.1.6 [3].** A family of subsets $\Im$ of a set $A$ is called a $\sigma$- algebra if :

i) $A \in \Im$,

ii) If $E \in \Im$, then $A - E \in \Im$, and

iii) If $E_i \in \Im$, i=1,2 ... ,m then $\bigcup_{i=1}^{\infty} E_i \in \Im$.

$(A, \Im)$ is called a measurable space, and $\Im$ are the measurable sets.

**Definition 1.1.7 [20].** A function $f : A \longrightarrow Y$ from a measurable space $(A, \Im)$ into a topological space$Y$ is called measurable if $f^{-1}(U)$ is a measurable set for every open set $U \subseteq Y$.

**Definition 1.1.8 [20].**

For an arbitrary random mapping

$$G : (A, \Im, P_{\Im}) \longrightarrow (R^l, L^l, P),$$

where $(A, \Im)$ is a measurable space, $\Im$ is $\sigma$- algebra on $A, P_{\Im}$ is a probability measure defined on $\Im, L^l$ is a Borel $\sigma$-algebra and $P$ is the probability measure induced by $G$ in $L^l$. The support of $G$ is defined as:

$$\text{supp } G = \{\tilde{a} \in R^l : p(\cup_{\varepsilon}(\tilde{a})) > 0 \ \forall \ \varepsilon > 0\} \ [20],$$

where $\cup_{\varepsilon}(\tilde{a})$ is the $\varepsilon -$ neighborhood of $\tilde{a}$.

**Theorem 1.1.1[16].** If $f(x_1, x_2, \ldots, x_n)$ is the probability density function of a collection of random variables $(\xi_1, \xi_2, \ldots, \xi_n)$, then the probability of the point $(\xi_1, \xi_2, \ldots, \xi_n)$ falling in the half-space $\xi_1 + \xi_2 + \cdots + \xi_n < x$ is:

$$\phi(x) = \int \cdots \int_{x_1 + x_2 + \cdots + x_n < x} f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n.$$

There are many types of probability distributions for describing various types of discrete and continuous random variables.Some of the common distributions are:

**(i) In the discrete case:**

1- Discrete Uniform distribution.

2- Binomial distribution.

3- Geometric distribution.

4- Poisson distribution.

5- Multinomial distribution.

6- Hypergeometric distribution.

7- Negative binomial (or Pascal) distribution.

**(ii) In the continuous case:**

1- Uniform distribution.

2- Normal distribution.

3- Gamma distribution.

4- Beta distribution.

5- Chi-Square distribution.

6- Bivariate Normal distribution.

7- Exponential distribution.

In this thesis, we deal with the parameters as random variables from normal distribution. So, we need to mention the density function of normally distributed variable $X$.

**Definition 1.1.9 [12].** A random variable $X$ is called a normally distributed with mean $\mu$ and variance $\sigma^2$, if its probability density function is on the form:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-1}{2}(\frac{x-\mu}{\sigma})^2}, \ -\infty < x < \infty.$$

Briefly, we write a normally distributed variable $X$, with mean $\mu$ and variance $\sigma^2$, as $N(\mu, \sigma^2)$. If $\mu = 0$ and $\sigma^2 = 1$, then the variable $z = \frac{x-\mu}{\sigma}$ is $N(0, 1)$ and is called standard normal distribution.

## 1.2 Stochastic Optimization Problems

Optimization is viewed as a kind of decision making in which information of the preferences among the alternatives is described by some utility (objective, performance,...) function that maps a set of feasible alternatives into the real numbers.

Stochastic optimization problems are the problems that involve uncertain values, say, uncertain variables or uncertain parameters.

There are many methods for modelling and solving stochastic optimization problems, (linear and nonlinear). We mention for example, the Chance-Constrained method [6] and the two-stages technique to treat a single objective stochastic problem [11].

### (a) The Chance-Constrained Method [6]

Chance-Constrained programming problem deals with situations in which the constraints appear in a probabilistic way, i.e. the probability that can be realized is less than one. Two treatments are stated in literature [6] to deal with the chance-constrained problems. The first one is the joint chance-constrained approach; which divides the constraints into groups and assigns one probability level for each group.

The second treatment of chance-constrained problems is the disjoint chance-constrained approach; which associates one probability level for each stochastic constraint in the problem. In the following, the disjoint chance-constrained approach is presented for the two possible forms of probabilistic constraints of a linear programming problem.
Consider the following chance-constrained problem:

$$\text{optimize} \qquad z = cx, \tag{1.2.1}$$

s.t

$$p(Ax \leq b) \geq \alpha \text{ .or } p(Ax \geq b) \geq \alpha, \tag{1.2.2}$$

$$0 < \alpha < 1, \tag{1.2.3}$$

$$x \geq 0, \tag{1.2.4}$$

where $c$ is a vector in $R^n$, $A$ is an $m \times n$ matrix, $b \in R^m$ with independent random components that have probability distribution functions $F_t(y) = p(b_t \leq y), t = 1, 2, \ldots, m$ and $\alpha$ is a vector represents pre-assigned measures chosen by the decision maker, such that $(1 - \alpha)$ represents the admissible risk that the random variables assume values such that: $Ax > b$
To find the deterministic constraint, the $t^{\underline{th}}$ probabilistic constraint takes one of the forms:

$$p\{(Ax)_t \leq b_t\} \text{ or } p\{(Ax)_t \geq b_t\}, \ (t = 1, 2, \ldots, m). \tag{1.2.5}$$

The deterministic equivalent constraint will be deduced in each of the two cases:

**Case 1:**

The probabilistic constraint
$p\{(Ax)_t \leq b_t\} \geq \alpha_t$.
is equivalent to:

$$1 - F_t(Ax)_t \geq \alpha_t,$$
$$F_t(Ax)_t \leq (1 - \alpha_t),$$
$$(Ax)_t \leq F_t^{-1}(1 - \alpha_t),$$

where $F_t^{-1}(*)$ is the inverse of the distribution function of $b_t$.

**Case 2:**

The probabilistic constraint
$p\{(Ax \geq b_t)\} \geq \alpha_t$.
is equivalent to:

$$F_t(Ax)_t \geq \alpha_t,$$
$$(Ax)_t \geq F_t^{-1}(\alpha_t).$$

**(b) The Two-Stage Technique [11]**

This technique was originated by Dantizg [11], and developed by Wets in [44]. It is used to obtain a deterministic program which is equivalent to a probabilistic single-objective linear programming problem. Although this technique enlarges the size of the problem, it is preferred to the chance-constrained method as it keeps the original

form of the decomposable problem. In the case that the right-hand-side of the model constitutes the random part, then the equivalent deterministic problem takes the primal block angular structure which can be solved using the decomposition technique [3].

Consider the linear program:

$$\text{optimize} \qquad z = cx, \tag{1.2.6}$$

Subject to

$$Ax = b, \tag{1.2.7}$$

$$Mx = R, \tag{1.2.8}$$

$$x \geq 0, \tag{1.2.9}$$

where $A$ is an $m \times n$ matrix and $M$ is an $s \times n$ matrix whose entries are constants, $c$ is $n-$vector, $b$ is $m-$vector and $R = (R_1, R_2, \ldots, R_r)^T$ is a vector of random variables.

The two-stage technique can be applied for this problem as:

**Stage1:**

Solve the problem without considering the random part $R$. i.e. solve the problem:

$$\text{optimize} \qquad z = cx, \tag{1.2.10}$$

Subject to

$$Ax = b, \tag{1.2.11}$$

$$x \geq 0. \tag{1.2.12}$$