SOME NEW LABELLINGS OF GRAPHS

A thesis

Submitted as
The requirements of the Ph.D. degree
In Pure Mathematics
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Submitted to
Mathematics Department
Faculty of Science
Ain Shams University
Cairo Egypt
2017

بسم الله الرحمن الرحيم

قالوا

سبحانك لا علم لنا

إلا ما علمتنا إنك أنت

العليم الحكيم

صدق الله العظيم

سورة البقرة الآية: ٣٢

Dedication

I dedicate my dissertation work to my mother and my father . I also dedicate this work to my beloved country Yemen.

Acknowledgements

In the name of Allah, great thanks and praise to Allah, who guided us to ideas of this work . Peace upon our master Mohammed the seal of prophets and messenger of Allah.

I would like to convey my deep respect my **supervisor Professor Mohammed Abd-El Azim Seoud***, for his advice, assistance and great support during this work. I also would like to thank my Co- supervisor* **Professor Ismail Mohammed Idris,** *for his encouragement and support during this work.*

I also would like to thank **Mr. M.S. Abo shady** *who did the algorithm written in C++ programming language.*

I also would like to offer my thanks and gratitude to my **family** *special feeling of gratitude to my loving* **parents and my uncle Ali,** *whose continuous support and encouragement kept fueling my passion for research in this field.* **My sisters and brothers** *have never left my side , they are very special dear to me and I would be grateful for them,* **especially Adeeb, Anas,** *and his* **daughter Noha***, whom I hope she finds an inspiration in it.*

Huda Jaber Mohammed

Abstract

In this thesis, we study: 1-sequentially additive, strongly *-, prime cordial , 3-equitable prime cordial, product cordial and k-prime graphs, and we give some properties of them. Also we give some of their families.

Keywords

Graph labeling**,** 1-sequentially additive labeling, strongly *-labeling, prime cordial labeling, 3-equitable prime cordial labeling, product cordial labeling and k- prime labeling.

Introduction

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Most graph labeling methods trace their origin to the one introduced by Rosa [26] in 1967, or the one given by Graham and Sloane [12] in 1980. Rosa [26] called a function f a $\beta-$ valuation of a graph G with q edges if f is an injection from the vertices of G to the set $\{0,1,\dots,q\}$ such that, when each edge xy is assigned the label $|f(x)-f(y)|$, the resulting edge labels are distinct. Golomb [11] subsequently called such labelings graceful and this is now the popular term. Rosa [26] introduced $\beta-$ valuations as well as a number of other labelings as tools for decomposing the complete graph into isomorphic subgraphs. In particular, $\beta-$ valuations originated as a means of attacking the conjecture of Ringel [25] that : K_{2n+1} can be decomposed into $2n+1$ subgraphs that are all isomorphic to a given tree with n edges. Although an unpublished result of Erdös says that most graphs are not graceful (see [12]), most graphs that have some sort of regularity of structure are graceful. Sheppard [32] has shown that there are exactly $q!$ gracefully labeled graphs with q edges. Rosa [26] has identified essentially three reasons why a graph fails to be graceful: (1) G has "too many vertices" and "not enough edges," (2) G "has too many edges," and (3) G "has the wrong parity". An infinite class of graphs that are not graceful for the second reason is given in [5]. As an example of the third reason Rosa [26] has shown that if every vertex has even degree and the number of edges is congruent to 1 or 2 (mod 4) then the graph is not graceful. In particular, the cycles C_{4n+1} and C_{4n+2} are not graceful. Harmonious graphs naturally arose in the study by Graham and Sloane [12] of modular versions of additive bases problems stemming from error-correcting codes. They defined a graph G with q edges to be harmonious if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x)+f(y)(mod\, q)$, the resulting edge labels are distinct. When G is a tree, exactly one label may be used on two vertices. They proved that almost all graphs are not harmonious. Analogous to the "parity" necessity condition for graceful graphs, Graham and Sloane [12] proved that if a harmonious graph has an even number of edges q and the degree of every vertex is divisible by 2^k then q is divisible by 2^{k+1}. Liu and Zhang [17] have generalized this condition as follows: if a harmonious graph with q edges has degree sequence $d_1, d_2, d_3, \ldots, d_p$ then $gcd(d_1, d_2, d_3, \ldots, d_p, q)$ divides $q(q-1)/2$. They have also proved that every graph is a subgraph of a harmonious graph.

Over the past four decades in excess of 1700 papers have spawned a bewildering array of graph labeling methods. Despite the unabated procession of papers, there are few general results on graph labelings. Indeed, the papers focus on particular classes of graphs and methods, and feature ad hoc arguments.

Labeled graphs serve as useful models for a broad range of applications such as: coding theory, xray crystallography, radar, astronomy, circuit design, communication network addressing, data base management, secret sharing schemes,and models for constraint programming over finite domains{see [6], [7], [23], [33], [35], [36], [40] and [19] for details. All graphs in this thesis are finite , simple and undirected. We shall use the basic notations and terominology in graph theory as in [8], [14],and [24], and in number theory as in [22].

Summary

This thesis consists of six chapters.

Chapter one:

In this chapter, some basic concepts and facts about number theory and graph theory which we need afterwards are introduced.

Chapter two:

This chapter, is devoted to study of 1-sequentially additive graphs. We determine all 1-sequentially additive graphs and not 1-sequentially additive of order 6 by using an algorithm written in C^{++} programming language. We label some 1- sequentially additive families. Some labelings are based on Skolem and hooked Skolem sequences.

Chapter three:

This chapter, is devoted to study of strongly *-graphs. We show that some special families of graphs are strongly *-graphs.

Chapter four:

This chapter, is devoted to study of prime cordial and 3-equitable prime cordial graphs. We give some properties of prime cordial /3-equitable prime cordial graphs, we determine all prime cordial graphs and non prime cordial graphs of order 7, we give some of their families, and we give the maximum number of edges in a 3-equitable prime cordial graph using two methods.

Chapter five:

This chapter , is devoted to study of product cordial graph. We give some properties of maximal product cordial graphs. We also give some necessary conditions for planar graphs to be product cordial, and we determine all product cordial planar graphs and non product cordial planar graphs of order at most 7.

Chapter six:

This chapter, is devoted to study of k- prime graphs. We give some properties of k- prime graphs and some families of k-prime graphs. Also necessary and sufficient condition for 4-prime graph is given, at the end, we determine all 4-prime graphs of order at most 6.

This thesis contains five papers:

1-The results of chapter 2 had been accepted to be published in the International Canadian Journal (Utilitas Mathematica) .

2-The results of chapter 3 are accepted to be published in the International Mathematical Forum.

3-The results of chapter 4 are submitted for publication and still under refereeing.

4- The results of chapter 5 were recently accepted to be published in the International Canadian Journal (Ars Combinatoria) .

5- The results of chapter 6 were recently accepted to be published in the International Canadian Journal (Ars Combinatoria).

Contents

Chapter one:

Background

1.1 Introduction

This chapter contains all basic concepts and facts that are needed in our thesis. In section 1.2: we give some definitions and theorems in number theory, in section 1.3: we give some definitions, examples, and theorems in graph theory .

1.2 Some definitions and theorems in number theory

The definitions and theorems of this section found in one of the following references, [15], [22].

Definition 1.2.1: Euler phi function $\varphi(k)$ is arithmetic function that counts the number of integers in the set $\{1,2,\dots,k-1\}$ that are relatively prime to k.

Example 1.2.2: $\varphi(2)=1, \varphi(3)=2, \varphi(4)=2, \varphi(5)=4, \varphi(6)=2$.

Definition 1.2.3: The arithmetic function $w(n)$counts the number of distinct prime divisors of the positive integer n, that is $w(n)=\sum_{p|n} 1$.

Example 1.2.4: $w(2)=1, w(3)=1, w(4)=1, w(5)=1, w(6)=2$.

Definition 1.2.5: $\pi(x)$ denotes the number of primes not exceeding x.

Example 1.2.6 : $\pi(2)=1, \pi(3)=2, \pi(4)=2, \pi(5)=3, \pi(6)=3$.

Theorem 1.2.7: Every positive integer can be written uniquely (up to order) as a product of prime numbers.

Example 1.2.8: We have the following positive numbers 4,5,6,7,can be written as follow $4 = 2^2,\ 5 = 5,\ 6 = 2.3, 7 = 7.$

Theorem 1.2.9 : (**The inclusion –exclusion principle)**

Suppose $n \in \mathbb{N}$, and A_i is a finite set for $1 \le i \le n$, it follows that

$$\left|\bigcup_{1\le i\le n} A_i\right| = \sum_{1\le i\le n} |A_i| - \sum_{1\le i<j\le n} \left|A_i \bigcap A_j\right| + \sum_{1\le i<j<k\le n} \left|A_i \bigcap A_j \bigcap A_k\right| - \cdots + (-1)^n \left|\bigcap_{1\le i\le n} A_i\right|$$

Example 1.2.10: Let $A = \{1,2,3,4\}, B = \{3,4,5,6\},\ C = \{2,3,4,5,6,7\}$, and we have by ***Theorem 1.2.9*** $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$, *i.e*, $7 = 4 + 4 + 6 - 2 - 3 - 4 + 2$, which is true.

1.3 Fundamental terminology in graph theory

The definitions and theorems of this section found in one of the following references, [8], [9],[10],[14].

Definition 1.3.1: A graph G is a finite non empty set V of objects called vertices (the singular is vertex) together with a set E of 2-element subsets of V called edges. We sometimes write $G = (V, E)$. Each edge $\{u, v\}$ of V is commonly denoted by uv or vu. If $e = uv$, then the edge e is said to join u and v.

Definition 1.3.2 : If uv is an edge of G, then u and v are adjacent vertices. Two adjacent vertices are referred to as neighbors of each other. If uv and vw distinct edges in G , then uv and vw are adjacent edges.

Definition 1.3.3: The vertex u and the edge uv are said to be incident with each other. Similarly v and uv are incident.

Definition 1.3.4: The degree of a vertex v in graph G is the number of vertices in G that are adjacent to v . Thus the degree of a vertex v is the number of the vertices in its neighborhood . A vertex of degree 0 is referred to as an isolated vertex and a vertex of degree 1 is an end vertex.

Example 1.3.5: In the following graph in *Figure 1* the vertices u and v are adjacent . The neighbors of the vertex u are v, x, and y. The edges vu and uy are adjacent. The vertex w is incident on the edge uw . The degree of the vertex u is 4, and the vertex z is an isolated

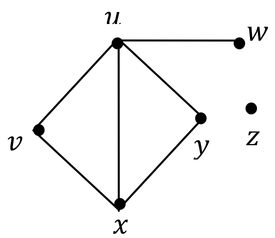

Figure 1.

Definition 1.3.6: A graph H is said to be a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$, then H is spanning subgraph of G.

Example 1.3.7: In *Figure* 2, G_1 is subgraph from G but G_2 is spanning subgraph of G .

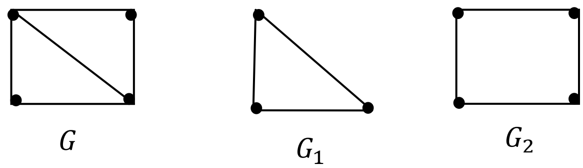

Figure 2.