On the Numerical Treatment of Elliptic System of Partial Differential Equations

Thesis by

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Submitted to

Department of Mathematics – Faculty of Science
Ain Shams University – Cairo – Egypt

In Partial Fulfillment of the Requirements
for the Degree of M. Sc. in Pure Mathematics

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2015

Acknowledge

First and foremost, Thanks are to **GOD**, the most beneficent merciful. I wish to express my deepest thanks and appreciation to my supervisor, Professor **Dr. Bayoumi Ibrahim Bayoumi** for suggesting the topics of this thesis, and for helpful guidance and constructive advice during the preparation and the supervision of this work, also grateful to **Dr. Ismail Kaoud Youssef**, for his effort and for his kind help and continuous encouragement, throughout the supervision of this thesis.

Finally, I am grateful to my family for their encouragement, their support and their help.

SUMMARY

Summary

Thesis title: "On the Numerical Treatment of Elliptic System of Partial Differential Equations".

The thesis introduces the numerical treatment of elliptic system of partial differential equations in the plane using the finite difference method. Two types of grids (square - triangular) are considered. The structure of the resultant algebraic system depends on the grid as well as the labeling of the grid points (the natural, the electronic, the RBG, and the spiral). Different forms of iterative methods (iterative methods without relaxation parameters, iterative methods with only one parameter, and iterative methods with two parameters) for solving linear algebraic systems are discussed.

This thesis consists of five chapters, Arabic summary, and English summary.

Chapter One: Elliptic Partial Differential Equations in the Plane

A system of two partial differential equations in the plane is studied. Transformation of systems of partial differential equations into canonical forms which contain the smallest possible number of parameters (only two parameters instead of twelve) is obtained. Classification of systems of partial differential equations in the plane is given. The finite difference method is used to transform differential equations into algebraic systems. The algebraic structures of the resultant algebraic systems as well as the grid labeling are established. The square grid is discussed with three labeling techniques (the natural, the electronic, and the red-black order) in connection with the standard Poission's problem.

Chapter Two: Variants of Successive Overrelaxation Techniques

Iterative methods for solving linear algebraic systems are established. Jacobi and Gauss Seidel methods are classified as iterative techniques without relaxation factors. SOR and KSOR methods are classified as iterative techniques with only one relaxation factor. Different forms of the modified successive overrelaxation methods (MSOR, MKSOR, MKSOR1, and MKSOR2) are introduced as iterative techniques with two relaxation factors. Iteration matrices and functional eigenvalue relations are given. Because of the fixed values of the spectral radii of Jacobi and Gauss Seidel iteration matrices, they are used to compare the convergence speeds with other methods and for the selection of relaxation parameters.

Chapter Three: Elliptic Systems Strongly Coupled Through the Mixed Derivative Term

Elliptic systems strongly coupled through the mixed derivative term are considered. Ordering of unknowns in connection with different labeling of grid points in the square grid is given. Selection of relaxation parameters which gives the smallest value of the spectral radius of the iteration matrix of different methods (SOR, KSOR, MSOR, MKSOR, MKSOR1, and MKSOR2) is established.

Chapter Four: Elliptic Systems Coupled Through the First Order Derivative Term

Elliptic systems coupled through the term of first order derivative with the effect of decoupling are studied. Ordering of unknowns in connection with different labeling of grid points in the square grid is given. Selection of relaxation parameters which gives the smallest value of the spectral radius of the iteration matrix for different methods (SOR, KSOR, MSOR, MKSOR, MKSOR1, and MKSOR2) is established. Reduction of the system to two uncoupled equations is considered. Application to a realistic physical problem is discussed.

Chapter Five: Boundary Value Problems on Triangular Domains

A treatment for the boundary value problem over a triangular grid is introduced similar to the treatment introduced by Young for the square grid. The effect of different labeling techniques (the natural, the electronic, the spiral, and red, black and green) on the structure of algebraic system is established. Elliptic equation with mixed derivative term is considered, a parameter r is introduced which enables one to obtain the results and the algebraic structures of elliptic problems without the mixed derivative term. Also, the system introduced in chapter four is discussed over the triangular grid.

It is worth to mention that:

- All calculations were done by the use Mathematica 8.0.
- The results of chapter five are published in the Journal of Applied and Computational Mathematics.

Contents

CHAPTER ONE

ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS IN THE PLANE

Chapter One
Elliptic Partial Differential Equations in the Plane

1.1 Introduction

A second order system of two partial differential equations, PDEs, in the plane can be written in the form

$$A\frac{\partial^2}{\partial x^2}\begin{bmatrix}u\\v\end{bmatrix}+2B\frac{\partial^2}{\partial x\partial y}\begin{bmatrix}u\\v\end{bmatrix}+C\frac{\partial^2}{\partial y^2}\begin{bmatrix}u\\v\end{bmatrix}+E\frac{\partial}{\partial x}\begin{bmatrix}u\\v\end{bmatrix}+F\frac{\partial}{\partial y}\begin{bmatrix}u\\v\end{bmatrix}=G(x,y), \qquad (1.1.1)$$

where u, v are real functions of x, y and $A, B, C, E,$ and F are $2x2$ real constant matrices. As in single equations the principal part (second order terms) controls the behavior of the system, so we consider systems of the form

$$A\frac{\partial^2}{\partial x^2}\begin{bmatrix}u\\v\end{bmatrix}+2B\frac{\partial^2}{\partial x\partial y}\begin{bmatrix}u\\v\end{bmatrix}+C\frac{\partial^2}{\partial y^2}\begin{bmatrix}u\\v\end{bmatrix}=0. \qquad (1.1.2)$$

There are twelve parameters in the system (1.1.2).The more parameters that system (1.1.2) involves, the more complicated it will be. It is well known that by means of linear transformation of independent variables, linear combination of equations, and linear transformation of unknown functions, system (1.1.2) can be transformed into an equivalent system. Moreover system (1.1.2) can be reduced to a canonical form which contains at most two independent parameters depending on the type of the system. Also, the parameters indicate the coupling of the system [12]. Decoupling, coupled system is a problem in itself. It is well known that solution of single equations is simple task in comparison of solution of system.

1.2 Equivalence between Systems of Partial Differential Equations [12]

The coefficient matrices in the system of PDEs (1.1.2) can be written in the form

$$A=\begin{bmatrix}a_1 & a_2\\a_3 & a_4\end{bmatrix}, B=\begin{bmatrix}b_1 & b_2\\b_3 & b_4\end{bmatrix}, \text{ and } C=\begin{bmatrix}c_1 & c_2\\c_3 & c_4\end{bmatrix}.$$

Similar to the transformations of independent variables in case of single PDE, also systems can be transformed into equivalent simpler systems. In case of systems there are three types of such transformations, linear combination of equations, transformation of independent variables, and moreover transformation of dependent variables. Let, the transformed system be

$$A_1\frac{\partial^2}{\partial x^2}\begin{bmatrix}u\\v\end{bmatrix}+2B_1\frac{\partial^2}{\partial x\partial y}\begin{bmatrix}u\\v\end{bmatrix}+C_1\frac{\partial^2}{\partial y^2}\begin{bmatrix}u\\v\end{bmatrix}=0, \qquad (1.2.1)$$

Operation (i) "Linear combination of equations"

The general formulation of the operation of linear combination of equations can be described in terms of multiplications of matrices. The coefficient matrices A_1, B_1, and C_1 of system (1.2.1) are obtained by multiplying A, B, and C on the left by a non-singular constant matrix

$$P = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix};\ |P| \neq 0, \tag{1.2.2}$$

i.e., $A_1 = PA, B_1 = PB$, and $C_1 = PC$.

Operation (ii) "Linear transformation of unknown functions"

$$\begin{bmatrix} u \\ v \end{bmatrix} = Q \begin{bmatrix} u_1 \\ v_1 \end{bmatrix};\ |Q| \neq 0, \tag{1.2.3}$$

where $Q = \begin{bmatrix} q_1 & q_2 \\ q_3 & q_4 \end{bmatrix}$ is a real non-singular constant matrix.

In this case the coefficients A_1, B_1 and C_1 of system (1.2.1) are obtained by multiplying A, B and C on the right by a non-singular matrix Q,

i.e., $A_1 = AQ,\ B_1 = BQ$, and $C_1 = CQ$.

Operation (iii) "Linear transformation of independent variables"

The operation of linear transformation of independent variables is the same as in case of single equation. Introduce the non-singular linear transformation

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix};\ \begin{vmatrix} p & q \\ r & s \end{vmatrix} \neq 0, \tag{1.2.4}$$

where p, q, r, and s are real constants. Since

$$\left.\begin{aligned} \frac{\partial^2}{\partial x^2} &\equiv p^2 \frac{\partial^2}{\partial x_1^2} + 2pr \frac{\partial^2}{\partial x_1 \partial y_1} + r^2 \frac{\partial^2}{\partial y_1^2}, \\ \frac{\partial^2}{\partial x \partial y} &\equiv pq \frac{\partial^2}{\partial x_1^2} + (ps + qr) \frac{\partial^2}{\partial x_1 \partial y_1} + rs \frac{\partial^2}{\partial y_1^2}, \\ \frac{\partial^2}{\partial y^2} &\equiv q^2 \frac{\partial^2}{\partial x_1^2} + 2qs \frac{\partial^2}{\partial x_1 \partial y_1} + s^2 \frac{\partial^2}{\partial y_1^2}. \end{aligned}\right\} \tag{1.2.5}$$

It is easy to verify that the coefficients of system (1.1.2) will be transformed into

$$\left.\begin{aligned} A_1 &= p^2 A + 2pqB + q^2 C, \\ B_1 &= prA + (ps + qr)B + qsC, \\ C_1 &= r^2 A + 2rsB + s^2 C. \end{aligned}\right\} \tag{1.2.6}$$

Definition (1.2.1)

The two systems (1.1.2) and (1.2.1) are said to be equivalent if one can be transformed into the other by means of successive applications of the three kinds of operation (i) - (iii).

Definition (1.2.2)
A system is said to be reducible if it is equivalent to a system with coefficient matrices

$$A_1 = \begin{bmatrix} a_1 & 0 \\ a_3 & a_4 \end{bmatrix}, B_1 = \begin{bmatrix} b_1 & 0 \\ b_3 & b_4 \end{bmatrix}, \text{ and } C_1 = \begin{bmatrix} c_1 & 0 \\ c_3 & c_4 \end{bmatrix}. \quad (1.2.7)$$

If system (1.1.2) is reducible, then with the aid of the three kinds of transformation (i) - (iii) it may be written in the form:

$$\left.\begin{aligned} a_1 \frac{\partial^2 u}{\partial x^2} + 2b_1 \frac{\partial^2 u}{\partial x \partial y} + c_1 \frac{\partial^2 u}{\partial y^2} = 0, \\ a_4 \frac{\partial^2 v}{\partial x^2} + 2b_4 \frac{\partial^2 v}{\partial x \partial y} + c_4 \frac{\partial^2 v}{\partial y^2} = -\left(a_3 \frac{\partial^2 u}{\partial x^2} + 2b_3 \frac{\partial^2 u}{\partial x \partial y} + c_3 \frac{\partial^2 u}{\partial y^2} \right). \end{aligned}\right\} \quad (1.2.8)$$

Hence the study of reducible system is equivalent to the study of two simple equations successively.

1.3 Classification of Biquadratic Forms

Classification of PDE is very important process, determining the type of the PDE equation Parabolic, Hyperbolic, or Elliptic illustrates the number and the nature of the associated auxiliary conditions, the behavior of solution, and controls the method of solution. Classification of systems of PDEs is a complicated process there are classification for systems of the form (1.1.2) which concentrates on second order terms only and define the so called biquadratic characteristic form introduced in [12]. Also, there are other classification techniques which considers all the terms of the differential operator [9, 11, 18].

Definition (1.3.1)
The biquadratic characteristic form of system (1.1.2) is defined as the determinant $F(\xi, \eta) = |A\xi^2 + 2B\xi\eta + C\eta^2|$.
The classification of the system (1.1.2) depends on the nature of the roots of the function $F(\xi, \eta)$.

The biquadratic form roots
The biquadratic form has nine different combinations of roots:
(1) Two distinct pairs of complex roots.
(2) A pair of double complex roots.
(3) A pair of complex roots and a double real root.
(4) A pair of complex roots and two distinct real roots.
(5) Four distinct real roots.
(6) Three distinct real roots.
(7) Two double real roots.
(8) A triple real roots and a simple real root.
(9) A quadruple real root.