A Numerical Treatment of Fractional Integro-Differential Equations

Thesis by

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Submitted to

Department of Mathematics, Faculty of Science
Ain Shams University, Cairo – Egypt

for the Degree of Doctor of Philosophy in Pure Mathematics

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Abstract

Thesis Title: **A Numerical Treatment of Fractional Integro–Differential Equations**

The thesis focuses on a numerical treatment of some differential and integral equations starting from the standard two-point boundary value problem (BVP) of the second order and its integral representation ending with the study of the general fractional order BVP. The main subject of the thesis is how to reduce the given problem to that of solving a system of algebraic equations. The thesis discusses three techniques for this treatment: the first is the finite difference method, the second is the Haar wavelet approach with collocation method and the third is the shifted Chebyshev polynomials of the first kind with collocation points. Also, the thesis considers the structure of the reduced algebraic system for the BVP and its integral form. The rates of convergence of the iterative technique are used for the comparison process. The thesis focuses on the three-part splitting technique for the coefficients matrix of the algebraic system. As a final conclusion of the thesis is using the second-degree Gauss-Siedle method for algebraic system obtained from the integral representation is better than using the Gauss-Siedle for the corresponding differential representation of the same problem. Also, a treatment for a system of integral equations using Haar wavelet approach is presented. Some numerical examples for all mentioned theoretical points were offered and the computational results were consistent with the theoretical studies.

Key words: Boundary value problems, Fredholm Integral equations, Finite difference method, Haar wavelet, Chebyshev Polynomials, Iterative techniques.

Summery

Summery

The significance of differential, integral and integro-differential equations is shown by the large number of the theoretical and practical applications that uses it. Although the topics of these equations are old, recent developments and the emergence of fractional boundary value problems and fractional integro-differential equations have re-added the importance of integral equations.

The accuracy of mathematical models using integral equations as compared to mathematical models using other methods is the global behavior. Mathematical models using the differential equations focus only on the local behavior. With developments the integro-differential equations was appeared. Since the main objective of the study of the differential or the integral equations is to obtain the function of the solution, the focus for a long time was on the analytical methods to find solutions to those equations. With the evolution of scientific calculations and the emergence of complexities in mathematical models, which restrict the use of analytical methods have emerged the importance of numerical methods.

Since the main objective is the numerical treatment of the linear fractional integro-differential equations using some numerical methods. Since the final stage in the numerical treatment is the solution of the reduced large linear systems of algebraic equations, then it is logical to study the properties of these large linear systems. In this thesis we use three numerical methods: the finite difference method, the collocation point method with wavelet functions especially (Haar wavelet) and the collocation point method with shifted Chebyshev polynomials of the first kind.

Numerous studies of the systems of algebraic equations resulting from the numerical treatment of differential equations have shown the properties of these systems, which are at most times tridiagonal, reflecting the local behavior of differential equations. In the study of integral equations, integro-differential equations and differential equations of fractional order, we observed that some of these properties were disappeared and the resulting systems were dense. When we try to solve the dense systems using the properties of the linear systems reduced from the corresponding equations in differential equations, we were exposed to the second degree iterative methods. The convergence rates of algebraic systems

resulting from differential equations and its integral form with the same approximation method were discussed.

In this thesis the numerical treatment of the integral representation of two-point Boundary Value Problems (BVP) especially that of second and fourth order are considered. The iterative techniques of the second degree are used in solving the resulting algebraic systems with comparison with the solution obtained using the same methods of the first degree. Also, the numerical solutions of the integral representation of BVP and integro-differential equations using Haar wavelet collocation method and comparison with other approximate methods are considered to show the efficiency of the Haar wavelet collocation approach. Besides, we introduce how to use the Haar wavelet collocation method to solve a system of linear integral equations. We construct a “fractional Green function” corresponding to the fractional boundary value problem (FBVP) which also gives an integral equation of Fredholm type. The methods of finite difference (shifted Grünwald–Letnikov) as well as the shifted Chebyshev collocation method are used to convert the fractional integro-differential equations and the integral form of FBVP into linear systems of algebraic equations with comparative analysis for solving these systems. The thesis consists of four chapters:

Chapter one: this chapter is devoted to the basic definitions and classifications of integral equations, integro-differential equations and system of integral equations are considered. The discretization techniques that transform the problem into linear systems of algebraic equations and methods for solving the resulting systems are mentioned. The concept of the wavelet approach (Haar wavelet) and how to use it in approximating the functions is presented. Also the basic definitions of the fractional integral and fractional derivative and methods for its discretization are introduced to use it in approximating the functions as well as in solving the given problem. The definition of the weighted residual methods, in particular the point collocation method, as well as the definition of the shifted Chebyshev spectral methods of the first kind, its derivative and its integration is presented.

Chapter two: in this chapter, the use of the three-part splitting concept of the coefficients matrix of the algebraic systems resulting from the solution of the integral equations, taking into account the algebraic construction of the coefficients matrix reduced from the corresponding BVP is the corner of stone. Since the iterative methods (Jacobi, Gauss-Seidle and SOR) can be compared with two-part splitting, and these methods are classified as iterative methods of first degree, the three-part splitting are accompanied by iterative methods of the second degree. We focused on second degree Gauss-Seidle iterative method and compared it with the same method of the first degree when solving the BVP and its integral form. Since the three-part splitting contains a parameter that can speed up the solutions, the optimal value of this parameter is calculated. Also, since in the three-part splitting each vector depends on the two previous ones, the best choice of the two initial vectors is presented. Then the convergence rate of the second degree Gauss-Seidle method and comparison with the convergence rate of the first degree Gauss-Seidle is shown. Three numerical examples are introduced to show the priority of the three-part splitting. The results were presented in the form of tables and figures to illustrate the solutions.

Chapter three: in this chapter one of the weighted residual methods is used with the wavelet functions, focusing on Haar wavelet functions and their integration. Haar wavelet with collocation method is used to convert the integral equation and its BVP form and the integro-differential equations into systems of algebraic equations. Also, the numerical treatment of system of linear integral equations using Haar wavelet approach with collocation method is introduced. We provide an algorithm to solve a system of linear integral equations using a colocation method with Haar wavelet bases functions. The technique is implemented on three numerical examples. The results were presented in the form of tables and figures to compare our solution with other semi-analytic methods such as (Differential transform method and Adomian decomposition method) to clarify the efficiency of the Haar wavelet collocation method.

Chapter four: in this chapter, the “fractional Green function” of the FBVP and its integral representation which also is of Fredholm kind are introduced. The shifted Grünwald–Letnikov method is used to convert the FBVP and its integral form as well as the fractional integro-differential equations to a system of algebraic equations. Another numerical technique, the shifted Chebyshev collocation method, is used to solve the integral representation of FBVP and fractional integro-differential equations. Numerical examples are used to compare between the FBVP and its integral form as well as the numerical comparison between the two numerical techniques are introduced.

All calculations are carried out with the help of the computer algebra program (Mathematica 11.1).

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