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Some Topological Properties of Soft Bitopological Spaces and Some of Its Applications

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(Pure Mathematics)

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Contents

Summary

The volume and complexity of the collected data in our modern society is growing rapidly. There often exist various types of uncertainties in those data related to complex problems in biology, economics, ecology, engineering, environmental science, medical science, social science, and many other fields. In order to describe and extract the useful information hidden in uncertain data, researchers in mathematics, computer science and related areas have proposed a number of theories such as probability theory [by Gerolamo Cardano in the sixteenth century], fuzzy set theory [106], Interval mathematics, intuitionistic fuzzy set theory [9] and rough set theory [84] which can be considered as mathematical tools for dealing with uncertainties. However, all of these theories have their own difficulties. Probability theory is applicable only for stochastically stable system. Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [105]. In response to this situation Zadeh [106] introduced fuzzy set theory as an alternative to probability theory. The most appropriate theory, for dealing with uncertainties is the theory of fuzzy sets developed. But, setting the membership function value is always been a problem in fuzzy set theory. Interval mathematics is not sufficiently adaptable for problems with different uncertainties. They cannot appropriately describe a smooth changing of information, unreliable, not adequate, and defective information, partially contradicting aims, and so on. Moreover, all these techniques lack in the parameterizations of the tools and they could not be applied successfully in tackling problems. In 1999, Molodtsov [79] initiated the novel concept of soft set as a new mathematical tool for dealing with uncertainties. He proposed a completely new approach for modeling vagueness and uncertainty in soft set theory. Soft set theory free from the difficulties affecting existing methods. Molodtsov [79] and Maji et al.[74] suggested that one reason for these difficulties is, possibly, the inadequacy of the parameterizations tool of the theory. There is no limited condition to the description of objects. Many of the established paradigms appear as special cases of soft set theory, so researchers can choose the form of parameters they need, which greatly simplifies

the decision making process and make the process more efficient in the absence of partial information.

In 2011, Shabir and Naz [97] initiated the study of soft topological spaces. They defined soft topology η on the collection of soft sets over X. Consequently, they defined basic notions of soft topological spaces such as open soft sets, closed soft sets, soft subspace, soft closure, soft neighborhood of a point, soft regular spaces, soft normal spaces and established their several properties. As a continuation of this, it is natural to investigate the behavior of topological structure or a combination of algebraic and topological structures in soft set theoretic form. Later, Çagman et al. [15], Zorlutuna et al.[109], Aygunoglu and Aygun [10], Hussain et al. [36], kandil et al. [[20], [45], [46], [47], [48], [49], [50], [51], [52], [53]] and Peyghan et al. [86] are continued to study the properties of soft topological space. They got many important results in soft topological spaces.

In 2011, Ittanagi [38] introduced the notion of soft bitopological space which is defined over an initial universal set X with fixed set of parameters E, also he introduced some types of soft separation axioms.

A study of soft bitopological spaces is a generalization of the study of soft topological spaces as every soft bitopological space (X, η_1, η_2, E) can be regarded as a soft topological space (X, η, E) if $\eta_1 = \eta_2 = \eta$.

Senel et al. [95] defined closed soft sets, α-closed soft sets, semi-closed soft sets, pre-closed soft sets, regular closed soft sets, g-closed soft sets and sg-closed soft sets on soft bitopological spaces. They also gave related properties of these soft sets and compared their properties with each other.

The main aim of this thesis can be summarized as follows:

- Introducing some concepts in soft bitopological spaces such as: pairwise open soft sets, pairwise closed soft sets, pairwise soft closure and pairwise soft interior, and studying the basic properties of them.
- Generating a supra soft topological space from a soft bitopological space which would enable us to study some of the characteristics of soft bitopological spaces through those spaces generated from it.
- Introducing new types of pairwise soft sets and generate soft topologies which have the property of Alexandroff, and studying some of the characteristics of soft bitopological spaces through these soft topologies generated.

- Extending the concept of the pairwise closed soft set to the generalized pairwise closed soft set, and studying some of their properties, as well as presenting the pairwise soft separation axioms and giving some characteristics and properties of these concepts.
- Studying some types of pairwise soft sets such as: the pairwise locally closed soft sets and pairwise λ-open soft sets, as well as introducing and studying the separation axioms associated with these types of soft sets.
- Introducing different types of pairwise soft continuous mappings of soft bitopological spaces, and studying many of their characterizations and the relationships between them.
- Introducing and studying the concept of soft connectedness in soft (ideal) bitopological spaces, and introducing some of their characterizations and properties.
- Giving some practical applications to the soft bitopological structures to solve some decision-making problems.

This thesis contains six chapters as follows:

Chapter one: It is the introductory chapter and contains the basic concepts and properties of topological spaces, bitopological spaces, soft sets, soft points, soft topological spaces, soft mappings, soft ideals and soft connectedness.

Chapter two: This chapter contains four sections as follows: **In section** (1), we generalized the notions related to soft topological spaces such as open soft sets, closed soft sets, soft interior, soft closure to the soft bitopological spaces. So, we introduce and study the notions of pairwise open (closed) soft sets, pairwise soft interior (respectively, closure) operator in soft bitopological space (X, η_1, η_2, E). The properties of these notions and some important results related to it are obtained. We show that the family of all pairwise open soft sets is a supra soft topology η_{12} which is containing η_1, η_2 but it is not soft topology in general. **In section** (2), we introduce and study the notion of pairwise soft kernel operator in soft bitopological spaces and we also introduce the notion of pairwise Λ-soft sets as soft sets that coincide with their pairwise soft kernel and we give some examples for support these notions. Moreover, we conclude that the family of all pairwise Λ-soft sets is an Alexandroff soft topology $\eta_{P\Lambda}$ which is finer than η_1, η_2 and containing η_{12}. **In section** (3), we define a new class of soft sets called pairwise λ-closed soft sets, and some of its properties are investigated. **In section** (4), we introduce the notion of pairwise soft sub kernel operator and we investigate some fundamental properties of this operator and we also introduce the notion of pairwise $\vee$-soft sets in a soft bitopological space (X, η_1, η_2, E) and we study the fundamental properties of pairwise $\vee$-soft sets and we investigate the associated soft topology $\eta_{p\vee}$.

Some results of this chapter have been published as follows:

- A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and Shawqi A. Hazza, Pairwise open (closed) soft sets in soft bitopological spaces, Ann. Fuzzy Math. Inform. 11 (4) (2016) 571−588 [57].

- A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and Shawqi A. Hazza, Some types of pairwise soft sets and the associated soft topologies, J. Intell. Fuzzy Syst. 32 (2) (2017) 1007−1018 [62].

Chapter three: This chapter contains six sections as follows: **In section** (1), we extend the concept of generalized closed soft sets to soft bitopological spaces called generalized pairwise closed soft sets [briefly, *gp*-closed soft sets] and we give some characterizations and properties of this notion. **In section** (2), we introduce and characterize the notion of pairwise soft $T_{\frac{1}{2}}$ [briefly, $PST_{\frac{1}{2}}$] in soft bitopological spaces and we provided some illustrative examples in support of the notion. **In section** (3), we introduce and characterize the notion of pairwise soft R_0 [briefly, PSR_0] in soft bitopological spaces and we provided some illustrative examples to support the notion. **In section** (4), we introduce the notions of generalized pairwise $\Lambda(\vee)$-soft sets and we investigate their basic properties. **In section** (5), we define a soft closure operator on the family of all generalized pairwise Λ-soft sets and generate, in usual manner, an Alexandroff soft topology $\eta_{gp\Lambda}$ on X which is finer than $\eta_{p\vee}$. Furthermore, we prove that $(X, \eta_{gp\Lambda}, E)$ is always a soft $T_{\frac{1}{2}}$ space. Moreover, we introduce characterization of pairwise soft $T_{\frac{1}{2}}$ by using generalized pairwise Λ-soft sets and we show that the concepts of generalized pairwise Λ-soft set and generalized pairwise closed soft set are independent concepts. **In section** (6), we define and study some soft separation (regularity) axioms in soft bitopological spaces in terms of pairwise softness namely, pairwise soft T_0, pairwise soft T_1, pairwise soft R_1, and pairwise soft T_2. Characterizations and properties of these soft separation axioms have been obtained. Moreover, we study the implications of these types of soft separation axioms in soft and crisp cases. Finally, we show that these properties are hereditary.

Some results of this chapter have been published as follows:

- A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and Shawqi A. Hazza, Generalized pairwise closed soft sets and the associate pairwise soft separation axioms, South Asian J. Math. 6 (2) (2016) 43−57 [54].

- A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and Shawqi A. Hazza, Some types of pairwise soft sets and the associated soft topologies, J. Intell. Fuzzy Syst. 32 (2) (2017) 1007−1018 [62].

- A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and Shawqi A. Hazza, Pairwise soft separation axioms in soft bitopological spaces, Ann. Fuzzy Math. Inform. 13 (5) (2017) 563−577 [59].

Chapter four: This chapter contains five sections as follows: **In section** (1), w continue to introduce some additional properties of $p\lambda$-closed soft sets. We also introduce and study a related new class of $PST_{\frac{1}{4}}$-spaces which lies between PST_0 and $PST_{\frac{1}{2}}$. Moreover, we show that there exists a very important relation between the notion of $p\lambda$-closed soft sets and the PST_i property, $i = 0, \frac{1}{4}, \frac{1}{2}$. **In sections** (2) **and** (3), we introduce and discuss new classes of soft sets namely; p-locally closed soft sets and $p\lambda$-open soft sets in soft bitopological spaces. We also conclude several important properties of such soft sets. In addition, we offer the notion of p-locally closed soft sets and we investigate a related new pairwise soft separation axiom PST_L which is independent from $PST_{\frac{1}{4}}$. The relationships between the $p\lambda$-closed soft sets and the p-locally closed soft sets are obtained. Furthermore, we introduce the notion of $p\lambda$-open soft sets and we construct supra soft topology associated with the class of $p\lambda$-open soft sets and we present pairwise soft separation axioms related to such soft sets namely; PST_λ. We studied the relationships between these types of separation axioms and the other in chapter 3. **In sections** (4) **and** (5), we introduce and study some new notions in soft bitopological spaces such as pairwise soft continuous mappings, $p\Lambda$ (resp. $p\lambda$, gp, pl)-soft continuous mappings, pairwise open (closed) soft mappings and pairwise soft homeomorphism mappings. Moreover, characterizations of these notions are obtained. In addition that, we investigate the behavior of pairwise soft separation axioms under these types of mappings. Furthermore, the relationships among these notions are studied.

Some results of this chapter have been published as follows:

- A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and Shawqi A. Hazza, On pairwise λ-open soft sets and pairwise locally closed soft sets, American Scientific Research Journal for Engineering Technology and Sciences 28 (1) (2017) 225−247 [56].

- A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and Shawqi A. Hazza, Some types of pairwise soft open (continuous) mappings and some related results, South Asian J. Math. 7 (2) (2017) 1−21 [61].

Chapter five: This chapter contains four sections as follows: **In section** (1), we introduce the notion of pairwise separated soft sets based on the soft space (X, η_{12}, E) which generate by a soft bitopological space (X, η_1, η_2, E) and study some of its properties. **In section** (2), Based on the notion of pairwise separated soft sets we introduce the notions of pairwise soft connected(disconnected) spaces and study some of their characterizations and properties. Also, we study the connected (disconnected) of soft sets in soft bitopological space (X, η_1, η_2, E) by using the supra soft topological space (X, η_{12}, E). Some examples have given to support these concepts. **In section** (3), we use the concept of soft ideal bitopological space $(X, \eta_1, \eta_2, E, \tilde{\mathcal{I}})$ to introduce the pairwise soft local function denoted by $()^{*\tilde{\mathcal{I}}}_{12}$. The properties of pairwise soft local function $()^{*\tilde{\mathcal{I}}}_{12}$ and some important results related to it have obtained. Moreover, we using the pairwise soft local function $()^{*\tilde{\mathcal{I}}}_{12}$ to generate the family η^*_{12} which is finer than η_{12} and we show that η^*_{12} is a supra soft topology but not a soft topology in general. Also, we introduce the notion of p^*-separated soft sets and study some of its properties. **In section** (4), based on the notion of p^*-separated soft sets we introduce the notion of p^*-soft connected(disconnected) spaces and study some of their characterizations and properties. In addition, we study the connected of soft sets in $(X, \eta_1, \eta_2, E, \tilde{\mathcal{I}})$ by using the supra soft topological space $(X, \eta_{12}, \tilde{\mathcal{I}}, E)$.

Some results of this chapter have been published as follows:

- A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and Shawqi A. Hazza, Pairwise soft connected in soft bitopological spaces, International Journal of Computer Applications 169 (11) (2017) 12−27 [58].

- A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and Shawqi A. Hazza, On pairwise soft connectedness, Mathematical Science Letters, 7 (2) (2018) 79−89 [55].

Chapter six: In this chapter, we introduce some applications of soft bitopological spaces in decision making.

Some results of this chapter have been published as follows:

- A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and Shawqi A. Hazza, Some applications of soft bitopological structures in decision making problems, Accepted [60].

0.1 Why soft set?

The real world is full of uncertainty (lack of information), vagueness (gradations in the notion of membership) and imprecision. Moreover, the great deal of data involved in economics, engineering, medical science and other fields are not always vivid and includes all kinds of uncertainty. In classical mathematics all the mathematical tools for modeling, reasoning and calculation are certain or precise which deals with certain problems. So that they cannot solve those complex problems in real life situations. In recent years researchers have become interested to deal with the complexity of uncertain data. There are a wide range of theories such probability theory, fuzzy set theory, intuitionistic fuzzy set theory, rough set theory, vague set theory and the interval mathematics which are considered as mathematical approaches to modeling vagueness. But each of these theories has its own inherent difficulties. Probability theory is applicable only for stochastically stable system. Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [105]. In response to this situation Zadeh [106] introduced fuzzy set theory as an alternative to probability theory. The most appropriate theory, for dealing with uncertainties is the theory of fuzzy sets developed. But, setting the membership function value is always been a problem in fuzzy set theory and their generalizations. Interval mathematics is not sufficiently adaptable for problems with different uncertainties. They cannot appropriately describe a smooth changing of information, unreliable, not adequate, and defective information, partially contradicting aims, and so on. The reason why there exist such difficulties is lack of the theory of expressing parameters. The tools for making sure parameters are so poor that uncertainty of parameters becomes the bottleneck of using these theories. To solve this problem, in 1999 Molodtsov set up the basic theory of soft sets which can well deal with uncertain, fuzzy and unclear information. This theory has proven useful in many different fields such as the smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory, and measurement theory. At present study on soft set is still discovering. Soft set theory is a new mathematical tool for dealing with uncertainties and is a set associated with parameters and has been applied in several directions. In particular, the work demonstrates that soft set theory can be applied to problems that contain uncertainties especially in decision making problems. These applications explain the voluminous work in this field within a short period of time. We emphasize that soft set has enough developed basic supporting structures through which various algebraic structures in theoretical point of view could be developed. The concept of soft set is fundamentally important in almost every scientific field. There are many problems that we may not be able to resolved by the fuzzy set theory but can be solved more precisely by the soft set theory.

0.2 What is soft set?

Let X be an initial universe set and E be a set of parameters. Let $P(X)$ denote the power set of X. A pair (G, E) is called a soft set over X, where G is a mapping given by $G : E \to P(X)$. In other words, a soft set over X is a parameterized family of subsets of the universe X. For $e \in E$, G(e) may be considered as the set of approximate elements of the soft set (G, E). As an illustration, let us consider the following example:

Example 0.2.1 *[79] Let $X = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ be the set of six houses under consideration, which Mr. Y is going to buy, and $E = \{e_1, e_2, e_3, e_4\}$ be the set of decision parameters which are stand for "expensive", "beautiful", "wooden" and "green surroundings", respectively.*

Suppose that:

$G(e_1) = \{h_2, h_4\}, G(e_2) = \{h_1, h_3\}, G(e_3) = \{h_3, h_4, h_5\}, G(e_4) = \{h_1, h_3, h_5\}$.

Thus, the soft set (G, E) is a parameterized family of subsets of the set X and give us a collections of approximate descriptions of an object.

For the purpose of storing a soft set in a computer, we could represent the soft set as a table in the following form.

Table 1: Tabular representation of a soft set (G, E)

X/E	$e_1(expensive)$	$e_2(beautiful)$	$e_3(wooden)$	$e_4(green\ surroundings)$
h_1	0	1	0	1
h_2	1	0	0	0
h_3	0	1	1	1
h_4	1	0	1	0
h_5	0	0	1	1
h_6	0	0	0	0

In this case, to define a soft set means to point out expensive houses, beautiful houses, and so on. It is worth noting that the sets $G(e)$ may be arbitrary. Some of them may be empty, some may have non-empty intersection.