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Some Studies on Proximity Spaces

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Contents

Summary

Proximity spaces characterize the proximity relation, a relation that provides a framework for identifying the nearness of a point to a set and the nearness of two sets, where nearness is based on the spatial relationship between objects. In 1952, The concept of proximity spaces was initiated by Efremovič [14]. Also, Lodato, and others [51, 52, 53, 62, 64] presented weaker axioms than those of Efremovič. In proximity spaces, a compatible proximity can be introduced only in completely regular topological spaces but in generalized proximities can in fact be introduced in any topological spaces. proximity relation is applied to solving problems based on human perception [65] that arise in areas such as digital images [66]. There are several published researches on proximity spaces some of them exist in [62]. Steiner [74] introduced the notion of S-quasi proximity. Hayashi [25] studied bitopological space generated by S-quasi proximity. Recently, Kandil et. al. [40, 41, 44] introduced a new approaches of proximity structure based on the ideal notion.

The notion of grill was initiated by choquet [11]. The grill is a powerful tool, since it related to many topics such as the theory of proximity spaces and the theory of compactifications etc. Thron [76] showed that the concept of grill plays an important role in the theory of proximities. Grills are extremely useful and convenient tool for many situations like filters and nets.

In 1965 Zadeh [79] introduced fuzzy set theory to solve uncertain problems related to economics, environment, engineering, society etc., also there are many theories which discuss uncertain problems such as: theory of rough sets, theory of probabilities etc. Chang [8] presented the notion of fuzzy topology. The notion of gradation of openness was introduced by Chattopadhyay et. al. [9]. Abd El-Monsef and Ramadan [1] introduced the notion of fuzzy supra topological spaces. Kandil et. al. [37] studied fuzzy bitopological spaces via the supra fuzzy topology generated from it.

Fuzzy proximity was studied by many authors [4, 34, 35, 45]. Ghanim et. al. [22] presented the concept of a gradation of S-quasi fuzzy proximity. Fuzzy supra proximity was studied by Kandil et. al. [38].

In 1999, Molodtsov [59] presented soft set theory as a new mathematical tool for studying uncertain problems and addressing the difficulties of the above theories. Maji et. el. [55] applied soft set theory in a decision making problems. In [59, 60] Molodtsov et. al. showed several applications of soft set theory such as smoothness of functions, game theory, operation research, Rieman integration, Perron integration, probability theory, measurement theory, etc. D. Pei and Miao [63] studied the relationships between soft sets and information system. There are several researches on soft set theory such as [10, 18, 19, 23, 33, 56].

Soft set theory has been applied in many topics like algebra, topology, etc. H. Aktas and N. Cagman [3] introduced the notion of soft group. Also E. İnan [30] introduced approximately semi group and ideals. Kharal and Ahmed [47], Majumdar and Samanta [58] introduced the notion of mapping of soft sets. several authors like Shabir and Naz [72], Hazara et. al. [26] studied the notion of soft topological spaces. Hussain and Ahmad [29] introduced some properties on soft topological spaces. Lashin et. al. [50] investigated rough set theory in the frame work of topological spaces. B. M. Ittanagi [31] introduced the notion of soft bitopological spaces which are defined over an initial universe with a fixed set of parameters. The notion of soft ideal was introduced by Kandil et. al. [39].

In 2014, Hazara et. al. [27, 28] introduced a different notion of basic proximity based on soft sets and the notion of soft proximity. Recently, Kandil et. al. [42, 43] introduced a new structure of proximity of soft sets and a new structure of soft proximity based on the ideal notion.

In 2001, Maji. et. al. [54] introduced fuzzy soft set theory which combines fuzzy sets and soft sets. Roy and Maji [67] applied fuzzy soft set theory in a decision making problems. Fuzzy soft set has been applied in many subjects like algebra, topology, etc. Aygunoglu and Aygun [6] studied fuzzy soft groups. Tany and Kandemir [75] presented the topological structure of fuzzy soft sets. Recently, there are several researches on fuzzy soft sets such as [20, 46, 73, 77]. Mukherjeel and Park [61] introduced the notion of fuzzy soft bitopological spaces. El-Sheikh [16] introduced the notion of gradation of openness of fuzzy soft topological spaces.

Cetkin et. al. [7] introduced the notion of soft fuzzy proximity. After that Demir and ozbakir [13] presented the fuzzy soft proximity structure in Katsaras's sense.

The main aims of this thesis can be summarized as follows:

1- Introducing a new structure of proximity spaces by using the notion of grill.

2- Presenting a new structure of basic proximity of soft sets based on the notion of soft ideal.

3- Introducing S-quasi proximity in soft setting and study the soft bitopological space generated by it.

4- Presenting the properties of supra fuzzy soft proximity spaces and the properties of fuzzy soft biproximity space via the associated supra fuzzy soft proximity space.

5- Studying the notion of gradation of S-quasi fuzzy soft proximity and study some of its properties.

This Thesis is constructed as follows:

Chapter 1: This chapter contains the necessary definitions, theorems and examples which are needed for the rest of the thesis.

Chapter 2: The aim of this chapter is to introduce a new structure of proximity relation by using the notion of grill. This type of proximity is a generalization of Efremovič [14] and (Leader [51], or Pervin [64], or Lodato [52]). So, for $\mathcal{G} = P(X)\backslash\{\phi\}$, we have the Efremovič proximity relation and the generalized proximity relation (Leader, or Pervin, or Lodato). Some of the results that we have obtained : every $\mathcal{G}$-normal T_1 space is $\mathcal{G}$-proximizable space. Also, for such space, we show that it has a unique compatible $\mathcal{G}$-proximity under the condition that X is compact relative to τ^*. Finally, for a surjective map $f : X \longrightarrow (Y, \delta_{f(\mathcal{G})})$ ($\mathcal{G}$ is a grill on X), we establish the largest $\mathcal{G}$-proximity $\delta_{\mathcal{G}}$ on X for which the map f is a $\mathcal{G}$-proximally continuous.

Some results of this chapter have been published as follows:

- A. Kandil, S. A. El-Sheikh, E. Said, $\mathcal{G}$-proximity spaces, Annals of Fuzzy Mathematics and Informatics, 14(6)(2017) 537-548.

- A. Kandil, S. A. El-Sheikh, E. Said, Generalized G-proximity spaces, South Asian Journal of Mathematics, 7(2)(2017) 140-147.

Chapter 3: A new approach of basic proximity relation of soft sets is introduced by using the notion of soft ideal. For $\tilde{I} = \{\tilde{\Phi}\}$, we have the basic proximity of soft sets which introduced by Hazara et. al. [28] and for other kinds of $\tilde{I}$ we obtain many kinds of basic proximity structures of soft sets. Also we reduce this structure by using the notion of soft ideals. Some results of these spaces are: if (X, τ, E) is $\tilde{I}$-soft normal space and (X, τ^*, E) is R_0'-space, then there exists $\tilde{I}$-Lodato proximity of soft sets $\delta_{\tilde{I}}$ such that $\tau^* = \tau_{\delta_{\tilde{I}}}$. Also the soft topology generated by $\tilde{I}$-basic proximity of soft sets is finer than the soft topology generated by R_0'-$\check{C}$ech closure operator of soft sets. Finally, for a bijective soft map $f : (X, E_1) \to (Y, \delta_{f(\tilde{I})}, E_2)$, we construct the largest $\tilde{I}$-Lodato proximity of soft sets $\delta_{\tilde{I}}$ on (X, E_1) such that f is $\tilde{I}$-proximally soft continuous mapping.

Some results of this chapter have been published as follows

- A. Kandil, S. A. El-Sheikh, E. Said, $\tilde{I}$-proximity spaces based on soft sets, Annals of Fuzzy Mathematics and Informatics, 15(1)(2018) 59-72.

Chapter 4: The notion of soft Steiner's quasi proximity relation is presented and the soft bitopological space generated by it is studied. Also, the relation between soft Steiner's quasi proximity relation and soft bitopological space is studied. Some of the results that we have obtained: A soft topological space (X, τ, E) has a soft S-quasi proximity δ such that $\tau = \tau_{C^\delta}$. Also, the soft bitopological space $(X, \tau_{C^\delta}, \tau_{K^\delta}, E)$ is pairwise zero dimensional, pairwise soft regular, pairwise soft normal and pairwise soft completely normal. Also the space (X, τ_{K^δ}, E) is soft quasi discrete space. Finally, A soft bitopological space (X, τ_1, τ_2, E) which is pairwise soft normal, pairwise soft T_1 and (X, τ_i, E) is soft quasi discrete space has a soft S-quasi proximity δ such that $\tau_j = \tau_{C^\delta}$ and $\tau_i = \tau_{K^\delta}$, $i, j = 1, 2$, $i \neq j$.

Some results of this chapter are:

- A. Kandil, S. A. El-Sheikh, E. Said, On soft S-quasi proximity spaces, (submitted).

Chapter 5: The purpose of this chapter is to present the notions of supra fuzzy soft topological space and supra fuzzy soft closure operator. Also, the notions of supra fuzzy soft proximity space and fuzzy soft biproximity space are presented. Finally, we study the properties of fuzzy soft biproximity space via the supra fuzzy soft proximity generated from it. some of the results that we have obtained: A space (X, τ, E) which is supra fuzzy soft normal and supra fuzzy soft T_1 has a supra fuzzy soft proximity δ such that $\tau_\delta = \tau$. Also, every proximally supra fuzzy soft continuous mapping is supra fuzzy soft continuous. Also, for every fuzzy soft bitopological space (X, τ_1, τ_2, E) which is P^*-normal and P^*T_1, the associated supra fuzzy soft topological space (X, τ_{12}, E) has a supra fuzzy soft proximity δ such that $\tau_\delta = \tau_{12}$.

Some results of this chapter are:

- A. Kandil, S. A. El-Sheikh, E. Said, Supra fuzzy soft topological spaces, supra fuzzy soft proximity and fuzzy soft biproximity spaces, (submitted).

Chapter 6: The aim of this chapter is to present the notion of gradation of Steiner's quasi fuzzy soft proximity space and study some of its properties. Also, we study the relation between fuzzy soft bitopological space (X, τ_1, τ_2, E), where τ_1 and τ_2 are two gradations of openness on (X, E) and (X, τ_2, E) is a fuzzy soft quasi discrete space, and a gradation of Steiner's quasi fuzzy soft proximity spaces. Finally, we introduce the notion of gradation of S-quasi fuzzy soft proximity mapping. Some of the results that we have obtained: every fuzzy soft topological space (X, τ, E) (where τ is a gradation of openness on (X, E)) has a gradation of S-quasi fuzzy soft proximity δ such that $\tau_{C^\delta} = \tau$. Also, every fuzzy soft quasi discrete space (X, τ, E) has a gradation of S-quasi fuzzy soft proximity δ such that $\tau_{K^\delta} = \tau$. Also, if (X, τ, τ^*, E) is a fuzzy soft bitopological space such that (X, τ_r, τ_r^*, E) is pairwise fuzzy soft normal and pairwise fuzzy soft T_1 for each $r \in (0, 1]$ and (X, τ^*, E) is a fuzzy soft quasi discrete space, then there exists a gradation of S-quasi fuzzy soft proximity δ such that $\tau = \tau_{C^\delta}$ and $\tau^* = \tau_{K^\delta}$. Finally, for a fuzzy soft mapping $\varphi : (X, E_1) \to (Y, \delta^*, E_2)$, we construct the greatest gradation of S-quasi fuzzy soft proximity δ on (X, E_1) which makes φ a gradation of S-quasi fuzzy soft proximity mapping.

Chapter 1

Preliminaries

In this section we recall some definitions, theorems and propositions which are needed in the sequel.

1.1 Topological spaces

Definition 1.1.1. [17] A non-empty collection $\tau \subseteq P(X)$ is called a topology on X if it satisfies the following axioms:

1. $X, \phi \in \tau$,
2. $A_1, A_2 \in \tau \Rightarrow A_1 \cap A_2 \in \tau$,
3. $A_i \in \tau \ \forall i \in I \Rightarrow \cup_i A_i \in \tau$.

The pair (X, τ) is called topological space. Every member of τ is called open set and its complement is called closed set. The family of all closed sets will be denoted by τ^c.

Definition 1.1.2. [17] Let (X, τ) be a topological space and $A \subseteq X$. Then, the closure of A, denoted by $cl(A)$ is defined by

$$cl(A) = \cap\{F \in \tau^c : A \subseteq F\}$$

Definition 1.1.3. [17] Let (X, τ) be a topological space and $A \subseteq X$. Then, the interior of A, denoted by $int(A)$ is defined by

$$int(A) = \cup\{F \in \tau : F \subseteq A\}$$

Definition 1.1.4. [48] Let X be a non-empty set. An operator $C : P(X) \to P(X)$ is called a Kuratowski closure operator if it satisfies the following axioms:

1. $C(\phi) = \phi$,
2. $A \subseteq C(A)$,
3. $C(A \cup B) = C(A) \cup C(B)$,
4. $C(C(A)) = C(A)$.

Theorem 1.1.5. *[48] Let X be a non-empty set and $C : P(X) \to P(X)$ is a Kuratowski closure operator. Then, there exists one and only one topology τ_c on X such that $C(A)$ is the τ_c-closure of the subset A of X, defined by:*

$$\tau_c = \{A \in P(X) : C(A^c) = A^c\}$$

Definition 1.1.6. [17] A topological space (X, τ) is called T_1-space if $\forall x, y \in X$ such that $x \neq y$, $\exists\ F, G \in \tau$ such that

$$x \in F, y \notin F \text{ and } y \in G, x \notin G$$

Definition 1.1.7. [17] A topological space (X, τ) is called normal space if $\forall F_1, F_2 \in \tau^c$ such that $F_1 \cap F_2 = \phi$, then $\exists\ H, G \in \tau$ such that

$$F_1 \subseteq H, F_2 \subseteq G \text{ and } H \cap G = \phi$$

Definition 1.1.8. [17] A topological space (X, τ) is called T_4-space if it is normal and T_1-space.

Definition 1.1.9. [17] A topological space (X, τ) is called R_0-space if one of the following equivalent conditions is satisfied

1. $x \in cl(\{y\}) \Leftrightarrow y \in cl(\{x\})$.
2. $x \in G \in \tau \Rightarrow cl(\{x\}) \subseteq G$.

1.2 Proximity spaces

Definition 1.2.1. [62] A binary relation $\delta \subseteq P(X) \times P(X)$ is said to be an Efremovič proximity if it satisfies the following conditions:

(p_1) $A\ \delta B \Rightarrow B\ \delta A$,

(p_2) $A\ \delta(B \cup C) \Leftrightarrow A\ \delta B$ or $A\ \delta C$,

(p_3) $A\ \delta B \Rightarrow A \neq \phi$ and $B \neq \phi$,

(p_4) $A \cap B \neq \phi \Rightarrow A\ \delta B$,

(p_5) $A\ \overline{\delta} B \Rightarrow$ there exist $C, D \subseteq X$ such that $A\ \overline{\delta} C^c$, $D^c\ \overline{\delta} B$ and $C \cap D = \phi$.

The pair (X, δ) is called proximity space. We will write $A\delta B$ if $(A, B) \in \delta$, otherwise we will write $A\overline{\delta}B$.

Lemma 1.2.2. *[62] If $A\delta B, A \subseteq C$ and $B \subseteq D$, then $C\delta D$.*

Theorem 1.2.3. *[62] Let (X, δ) be a proximity space. Then the operator*

$$\delta : P(X) \to P(X)$$

defined by

$$A^\delta = \{x : x\delta A\}$$

is a Kuratowski closure operator and generates a topology on X called τ_δ given by

$$\tau_\delta = \{A \subseteq X : A^{c^\delta} = A^c\}$$

Lemma 1.2.4. *[62] For subsets A and B of a proximity space (X, δ),*

$$A\delta B \Leftrightarrow A^\delta \delta B^\delta$$

Definition 1.2.5. [62] If (X, τ) is a topological space and there is a proximity δ such that $\tau = \tau_\delta$, then τ and δ are said to be compatible.

Theorem 1.2.6. *[62] In a τ_4-space (X, τ), the relation δ defined by:*

$$A\delta B \Leftrightarrow cl(A) \cap cl(B) \neq \phi$$

is a compatible proximity, where $cl(A)$ and $cl(B)$ are the τ-closure of A and τ-closure of B, respectively.

Definition 1.2.7. [62] A subset B of a proximity space (X, δ) is a δ-neighbourhood of H (denoted by $H \ll B$) iff $H\bar{\delta}B^c$.

Lemma 1.2.8. *[62] Let (X, δ) be a proximity space. Then*

(i) $A \ll B$ *implies* $A^\delta \ll B$,

(ii) $A \ll B$ *implies* $A \ll int^\delta(B)$.

where A^δ and $int^\delta(B)$ denotes respectively, the closure and the interior of A and B in τ_δ.
Thus $A \subseteq int^\delta(B)$, implies that a δ-neighbourhood is a topological neighbourhood.

Theorem 1.2.9. *[62] Let (X, δ) be a proximity space. Then the relation $\ll$ satisfies the following properties:*

(1) $X \ll X$,

(2) $A \ll B$ *implies that* $A \subseteq B$,

(3) $A \subseteq B \ll C \subseteq D$ *implies that* $A \ll D$,

(4) $A \ll B_i$ *for* $i = 1, ..., n$ *iff* $A \ll \cap_{i=1}^n B_i$,

(5) $A \ll B$ *implies that* $B^c \ll A^c$,

(6) $A \ll B$ *implies that* $\exists\ C$ *such that* $A \ll C \ll B$,

If δ is a separated proximity, then

(7) $x \ll y^c$ *iff* $x \neq y$.

Corollary 1.2.10. *[62] $A_i \ll B_i$ for $i = 1, ..., n$ implies that*

$$\cap_{i=1}^n A_i \ll \cap_{i=1}^n B_i \text{ and } \cup_{i=1}^n A_i \ll \cup_{i=1}^n B_i.$$