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Summary

A bitopological space (X, τ_1, τ_2) was introduced by Kelly [38] in 1963, as a method of generalizes topological spaces (X, τ). Every bitopological space (X, τ_1, τ_2) can be regarded as a topological space (X, τ) if $\tau_1 = \tau_2 = \tau$. Furthermore, he extended some of the standard results of separation axioms and mappings in a topological space to a bitopological space. The notion of connectedness in bitopological spaces has been studied by Pervin [51], Reily [52] and Swart [58].

In 1983 Mashhour et al. [42] introduced the notion of supra topological spaces by dropping only the intersection condition. Kandil et al. [35] generated a supra topological space (X, τ_{12}) from the bitopological space (X, τ_1, τ_2) and studied some properties of the space (X, τ_1, τ_2) via properties of the associated space (X, τ_{12}). Thereafter, a large number of papers have been written to generalize topological concepts to bitopological setting [15, 18, 20, 19, 35, 55].

Generalized open sets play a very important role in general topology and they are now the research topics of many topologists worldwide. Andrijevic [5, 4] introduced a class of generalized open sets in a topological space as b-open sets and α-sets. The class of b-open sets is contained in the class of β-open sets and contains both semi-open sets and pre-open sets. Levine [39], Mashhour et al. [41], Njastad [49] and Abd El-Monsef et al. [2] introduced semi-open sets, pre-open sets, α-open sets and β-open sets, respectively.

A. Császár [13, 14] in 2002 introduced the concepts of generalized neighbourhood systems and generalized topological spaces. He also introduced the concepts of continuous functions and associated interior and closure operators on generalized neighbourhood systems and generalized topological spaces. In particular, he investigated characterizations for the generalized continuous function by using a closure operator defined on generalized neighbourhood systems. C. Dungthaisong et al [11] introduced the concept of bigeneralized topological spaces and studied (m, n)- closed sets, (m, n)- open sets and (m, n)- generalized closed sets in bigeneralized topological spaces.

Thivagar [59]in 2013 introduced the concept of nano topological spaces with respect to a subset X of a universe V. He studied the relationships between some near nano open sets in nano topological spaces[59, 60, 61]. A nano topology which is named so due to its size, because it can have only a maximum of five elements in it. The concept of nano bitopolgical space presented by a number of researchers in different ways like Bhuvaneswari, Rasya Banu and Nirmala Rebecca Paul[8, 9, 48]. The great importance of nano topological space and nano bitopological space are its applications that touch many areas of life.

This thesis is devoted to

1. Introduce generalized closed sets in bitopological spaces.
2. Introduce generalized locally pairwise closed sets on bitopological spaces and study some of its properties.
3. Introduce generalized pairwise star separation axiom on bitopological spaces.
4. Introduce gp-connectedness, lp-connectedness glp, glp^* and glp^{**}- connectedness on bitopological spaces.
5. Study some properties in bigeneralized topological spaces.
6. Introduce supra uniform space, biuniform space and study some of its properties.
7. Introduce New class of generalized pairwise closed set in nano bitoplogical spaces and new class of generalized locally pairwise closed set in nano bitoplogical spaces.
8. Introduce an application on psychological scales based on nano topological space.
9. Study two applications in medicine filed based onby nano bitopological space.

This thesis contains 7 chapters:-

Chapter 1 is the introductory chapter, so it contains the basic concepts and properties of topological space such as closure, interior, boundary, functions, separation axioms. It is also containing basic concepts and properties of uniform spaces and supra topological space and we using that it's properties to introduced notions in more sections. The basic concepts and properties of bitopological spaces and bigeneralized topological spaces are presented. Further, this chapter contains the basic notions of nano topological space and nano bitopological space which are needed in sequel.

In Chapter 2, we introduce the concept of generalized pairwise closed (open) sets in bitoplogical spaces and we study some of their properties. Also, we introduce the notion of generalize pairwise closed (open) sets via ideals, study some properties and investigate the relation between two approaches. We also introduce the notion of generalized locally pairwise closed sets, generalized locally pairwise closed star and generalized locally pairwise closed star star in bitopological spaces and study some of their properties. We have used the previous concepts to define and study the concept of pairwise- submaximal and generalized pairwise submaximal P- submaximal. Finally,we study the concept of generalized locally pairwise closed ($GLPC$-) functions and some of their properties.

Some results of this chapter are:

- "A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and E. A. Shalaby, Generalized closed sets in bitopological spaces, South Asian J. Math. 6(2) (2016), 72−81."
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The goal of Chapter 3 is to introduce the concepts of generalized pairwise star T_0 space, generalized pairwise star T_1 space, generalized pairwise star T_2 space, generalized pairwise star T_3 space, generalized pairwise star T_4 space, generalized pairwise star R_0 space and generalized pairwise star R_1 space, we also study some of their properties.

The results of this chapter is:

- "A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and E. A. Shalaby, Generalized pairwise star separation axiom in bitopological spaces, *international journal of mathematical archive* 9(6) (2018) 67-74.

In Chapter 4, we used the notions of generalized pairwise open set, locally pairwise open set and generalized locally pairwise open set on bitopological spaces to introduce the notion of *gp*- connectedness *lp*- connectedness and *glp*- connectedness and study some of their properties. The properties of the space (X, τ_1, τ_2) are studied through the study of the space (X, τ_{12}) which is a supra topology associated to the bitopological space (X, τ_1, τ_2).

In Chapter 5 we generate a generalized topological space (X, μ_{12}) from the bigeneralized topological space (X, μ_1, μ_2). We study some properties of (X, μ_1, μ_2), like, operators, generalized continuity, separation axioms, generalized closed sets and boundary of set, via generalized space (X, μ_{12}). The relations between the current study and the previous one have obtained. The important of this approach that, we dealing with one family instead of two families μ_1, μ_2 and this approach is a generalization of the results in (X, μ_1, μ_2) and in $(X, \mu_i), i = 1, 2$.

The results of this chapter is:

- "A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and E. A. Shalaby, Some Properties in Bigeneralized Topological Spaces , *South Asian J. Math.*'accepted'

In Chapter 6 we defined the notion of supra uniform space and study some of their properties. Also, we introduce the notion of biuniform space and study some properties.

In Chapter 7 we find the relation between uniform topological space which is generated by uniform space with the same equivalent relation as a base for uniform space and nano topology which is generating by the same equivalent relation. Also, we make an application on nano topological space in Psychological scales and we used $WINSTEPS$ which is one of important IRT programs to comparing between the result which given by it and the result we get it from nano topological space.
Also, we define the concept of nano supra topological space $\tau_{R_1R_2}(X_1X_2)$ which is associated by nano bitopolgical space $(V, \tau_{R_1}(X_1), \tau_{R_2}(X_2))$ and we introduce the concept nano pairwise generalized closed(open) sets, nano generalized locally pairwise closed sets,nano generalized locally pairwise closed star and nano generalized locally pairwise closed star star in nano bitopological space $(V, \tau_{R_1}(X_1), \tau_{R_2}(X_2))$ by the new space $(V, \tau_{R_1R_2}(X_1X_2))$ which is nano supra topological space associated by nano bitopological space $(V, \tau_{R_1}(X_1), \tau_{R_2}(X_2))$ and we study some of their properties. Finally, we introduced two applications in the field of medicine by using nano bitopological spaces.

The results of this chapter is:

- "A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and E. A. Shalaby, Nano topological spaces: An application on psychological scales, *Appl. Math. Inf. Sci.* accepted."
- "A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and E. A. Shalaby, New class of generalized pairwise closed set in nano bitoplogical spaces and application, . submitted."

Chapter 1

Preliminaries

The purpose of this chapter is to present a short survey of some needed definitions and theories of the material used in this thesis.

1.1 Some basic concepts of topological structures

The aim of this section is to collect the relevant definitions and results from topology about interior, closure, boundary, separation axioms and mappings.

Definition 1.1.1 *[22] Let X be a non empty set. A class τ of subsets of X is called a topology on X if it satisfies the following axioms:*

1. *$X, \phi \in \tau$,*
2. *An arbitrary union of the members of τ is in τ,*
3. *The intersection of any two sets in τ is in τ.*

The members of τ are then called τ-open sets, or simply open sets. The pair (X, τ) is called a topological space. A subset A of a topological space (X, τ) is called a closed set if its complement A^c is an open set. If τ satisfies the conditions 1 and 2 only, then τ is said to be a supra topology on X and the pair (X, τ) is called a supra topological space [42].

Definition 1.1.2 *[46] Let (X, τ) be a topological space and $A \subseteq X$. Then,*

1. *$cl(A) = \cap\{F \subseteq X : A \subseteq F$ and F is closed$\}$ is called the τ-closure of A,*

2. *$int(A) = \cup\{G \subseteq X : G \subseteq A$ and G is open$\}$ is called the τ-interior of A,*

3. *$b(A) = cl(A) \setminus int(A)$ is called the τ-boundary of A.*

Definition 1.1.3 *[46] Let (X, τ) be a topological space and $x \in X$ be an arbitrary point. A set $N \subseteq X$ is called a neighborhood of x if $x \in int(N)$, or equivalently, if there exists an open set U such that $x \in U \subseteq N$.*

Definition 1.1.4 *[25] Two subsets A and B of a topological space (X, τ) are said to be separated from each other in X if and only if $clA \cap B = A \cap clB = \phi$.*

Definition 1.1.5 *[25] A topological space (X, τ) is called:*

1. *a T_o-space if for every pair of distinct points of X there is a neighbourhood of at least one to which the other does not belong,*

2. *a T_1-space if for every pair of distinct points of X there exists a neighbourhood of each to which the other does not belong,*

3. *a T_2-space if for every pair of distinct points x and y of X there exist disjoint neighbourhoods of x and y,*

4. *a regular space if for each closed subset B of X and each point x in X such that $x \notin B$ there exist disjoint open neighbourhoods of B and x,*

5. *a T_3-space if it is both a regular space and a T_1-space,*

6. *a normal space if and only if every pair of disjoint closed subsets A and B of X have disjoint open neighbourhoods,*

7. *a T_4-space if it is both a normal space and a T_1-space,*

8. *a completely normal space if every pair of subsets A and B of X separated from each other have disjoint neighbourhoods,*

Theorem 1.1.1 *[25] A finite subset of a T_1-space is closed. Also, a finite T_1-space is discrete.*

Definition 1.1.6 *Let (X, τ) be a topological space and $A \subseteq X$. Then, A is said to be:*

1. *A semi-open set if $A \subseteq cl(int(A))$ [40],*

2. *A pre-open set if* $A \subseteq int(cl(A))$ *[41],*

3. *An* α*-open set if* $A \subseteq int(cl(int(A)))$ *[49],*

4. *A* β*-open set if* $A \subseteq cl(int(cl(A)))$ *[1],*

5. *A b-open set if* $A \subseteq cl(int(A)) \cup int(cl(A))$ *[4],*

6. *A generalized closed (g-closed, for short) set if* $cl(A) \subseteq O$ *whenever* $A \subseteq O$ *and* O *is an open set [39].*

Definition 1.1.7 *[22] A function* $f : (X, \tau) \to (Y, \theta)$ *is called:*

1. *Continuous if the inverse image of every open subset of* Y *is an open subset of* X*,*

2. *Open (respectively closed) if the image of every open (respectively closed) subset of* X *is an open (respectively closed) subset of* Y*,*

3. *homeomorphism if* f *is one-to-one correspondence, continuous and open.*

Definition 1.1.8 *[46] A class* $\{A_i : i \in I\}$ *of sets is said to have the finite intersection property if every finite subclass* $\{A_{i_1}, ..., A_{i_m}\}$ *has a non-empty intersection, i.e.,* $A_{i_1} \cap ... \cap A_{i_m} \neq \phi$*.*

Proposition 1.1.1 *[39]. If* A *and* B *are g-closed sets in space* (X, τ)*, then* $A \cup B$ *is a g-closed set in* X*.*

Definition 1.1.9 *[30]. An ideal* I *on a set* X *is a non empty collection of subset of* X *(i.e* $I \subseteq P(X)$*) such that*

1. *if* $A \in I$ *and* $B \subseteq A$*, then* $B \in I$*,*

2. *if* $A, B \in I$*, then* $A \cup B \in I$*.*

Definition 1.1.10 *[37] A non empty collection* ϑ *of subsets* $U \subseteq X \times X$ *is called a uniformity structure on* X *if it satisfies the following axioms:*

1. *If* $U \in \vartheta$*, then* $\Delta \subseteq U$*, where* $\Delta = \{(x, x) : x \in X\}$ *is the diagonal on* $X \times X$*.*

2. *If* $U \in \vartheta$ *and* $U \subseteq V$ *for* $V \subseteq X \times X$*, then* $V \in \vartheta$*.*

3. *If $U \in \vartheta$ and $V \in \vartheta$, then $U \cap V \in \vartheta$.*

4. *If $U \in \vartheta$, then there is $V \in \vartheta$ such that $V \circ V \subseteq U$, where $V \circ V$ denotes the composite of V with itself. (The composite of two subsets V and U of $X \times X$ is defined by, $V \circ U = \{(x,z) : \exists y \in X : (x,y) \in U \wedge (y,z) \in V\}$.*

5. *If $U \in \vartheta$, then $U^{-1} \in \vartheta$, where $U^{-1} = \{(y,x) : (x,y) \in U\}$ is the inverse of U.*

Properties (2) and (3) state that ϑ is a filter. If the last property is omitted we call the space quasi uniform. The elements U of ϑ are called entourages from the French word for surroundings. One usually writes $U[x] = \{y : (x,y) \in U\}$.

Definition 1.1.11 *[37, 33] Let X be a non- empty set. If $\beta \subseteq P(X \times X)$ and satisfied the conditions:*

1. *$\Delta \subseteq U$, $\forall U \in \beta$.*

2. *If $U \in \beta$ then, $U^{-1} \in \beta$.*

3. *If $U \in \vartheta$ and $V \in \vartheta$, then $U \cap V \in \vartheta$.*

4. *If $U \in \beta$, then $\exists V \in \beta$ such that $V \circ V \subseteq U$.*

then β is said to be a base of uniform space. The uniformity ϑ_β generated by β is the collection of all super sets of members of β and $\vartheta_\beta = \{U : U \in P(X \times X), \exists V \in \beta$ such that $V \subseteq U\}$.

Example 1.1.1 *For any non empty set X we have,*

1. *$\Upsilon = \{U : \Delta \subseteq U, U \subseteq X \times X\}$ is a largest uniformity on X.*

2. *$\vartheta_0 = \{X \times X\}$ is the smallest uniformity on X.*

3. *If $|X|= 1$, then the largest and smallest uniformity is equal.*
   If $|X|= 2$, then the largest and smallest uniformity are the only possible supra uniformities on X.

Proposition 1.1.2 *[37] [33] Let X be a non empty subset and $\Delta \subseteq R \subseteq X \times X$. If R is symmetric and $R \circ R = R$ (in other words R is equivalence relation on X) ,then the family of all super sets of R is a uniformity on X.*