بسم الله الرحمن الرحيم

يا أيها الذين ءامنوا اصبروا
وصابروا ورابطوا واتقوا الله
لعلكم تفلحون

صدق الله العظيم

I dedicate this thesis to
My parents, My husband ”Ahmed”, My draughters ”Menna and Mariam”, My sister ”Nora” *and* ***My brothers ”Ahmed and sheiref”.***

On Stability of Difference and Dynamic Equations

Thesis Submitted for the Ph. D. degree

By

Samah Mohammed Ahmed

M. Sc, Faculty of Science (girls branch),

Mathematics department, Al-Azhar University

Under supervision of

Prof. Dr. Afaf S. Zaghrout

Math. Dept. - Faculty of Science

Al-Azhar Univ. and the director of

Computer Center in Al-Azhar Univ.

Prof. Dr. Alaa E. Hamza

Mathematics Department

Faculty of Science

Cairo University

Dr. Fatma Hassan Mohammed

Mathematics Department

Faculty of Science

Al-Azhar University

Dr. Ahmed Morsy

Basic Science Dept.

Faculty of Engineering

BUE

(2014)

APROVAL SHEET FOR SUBMISSION

Thesis Title: **On Stability of Difference and Dynamic Equations**.

Name of candidate: **Samah Mohammed Ahmed** .

This thesis has been approved for submission by the supervisors:

1- **Prof. Dr. Afaf S. Zaghrout**

Mathematics Department, Faculty of Science, Al-Azhar University.

Signature:

2- **Prof. Dr. Alaa E. Hamza**

Mathematics Department, Faculty of Science, Cairo University.

Signature:

3- **Dr. Fatma Hassan Mohammed**

Mathematics Department, Faculty of Science, Al-Azhar University.

Signature:

4- **Dr. Ahmed Morsy**

Basic Science Department, Faculty of Engineering, BUE.

Signature:

Prof. Dr. Hoda Abd-Eldaim

Chairman of Mathematics Department

Faculty of Science - Al-Azhar University

I dedicate this thesis to

My parents, My husband "Ahmed", My daughters "Menna and Mariam", My sister "Nora" *and* ***My brothers "Ahmed and sheiref".***

Table of Contents

Acknowledgments

First and foremost, I would like to thank **ALLAH** for helping me to complete this research study.

I would like to grab this opportunity to express my gratitude to several people who have been very supportive and of great help throughout my Ph. D. study. First of all, I would like to thank my supervisor **Prof. Dr. Afaf S. Zaghrout** for her valuable advice and great guidance throughout my study. Special thanks go to my supervisor **Prof. Dr. Alaa E. Hamza** for his patience, guidance and encouragement without which I would have never been able to finish my thesis now. I also would like to express my appreciation to **Dr. Fatma Hassan** and **Dr. Ahmed Morsy** for their advice and enourgement. Thanks due to **Prof. Dr. Hoda Abd Eldaim**, Head of Mathematics Department, Faculty of Science, Al-Azhar University, for her continuous encouragement. So, I am very grateful to the Mathematics Department, Faculty of Science, Al-Azhar University for offering me a Ph. D. position.

Also, I am very grateful to the members of Mathematics Department, Faculty of Science, Cairo University for sharing their research experience and knowledge.

Finally, I am greatly indebted to my parents, my husband **Ahmed**, my daughters **Menna and Mariam**, sister and brothers for their constant support, understanding and encouragement.

Abstract

Thesis Title: **On Stability of Difference and Dynamic Equations.**

Named of candidate: **Samah Mohammed Ahmed.**

In this thesis,

- We present new results of the calculus based on the Hahn operator.
- We establish an existence and uniqueness result of solutions of Hahn difference equations by using the method of successive approximations.
- We establish the theory of linear Hahn difference equations.
- We establish characterizations of many types of stability of linear Hahn difference equations of the form $D_{q,\omega}x(t) = p(t)x(t) + f(t)$.

Supervisors:

1- Prof. Dr. Afaf S. Zaghrout

2- Prof. Dr. Alaa E. Hamza

3- Dr. Ahmed Morsy Abd Allah

4- Dr. Fatma Hassan Mohammed.

Introduction

Hahn introduced his difference operator which is defined by

$$D_{q,\omega}f(t) = \frac{f(qt+\omega) - f(t)}{t(q-1)+\omega},$$

where $0 < q < 1$ and $\omega > 0$ are fixed real numbers [20, 21]. This operator unifies and generalizes two well-known difference operators. The first is Jackson $q-$ difference operator defined by

$$D_q f(t) = \frac{f(qt) - f(t)}{t(q-1)},$$

where q is fixed. Here f is supposed to be defined on a $q-$ geometric set $A \subset \mathbb{R}$ for which $qt \in A$ whenever $t \in A$, see [2, 6, 7, 13, 14]. The second operator is the forward difference operator

$$\triangle_\omega\, f(t) = \frac{f(t+\omega) - f(t)}{\omega},$$

where $\omega > 0$ is fixed, see [8, 9, 29, 30]. It is worth mentioning that

$$\begin{aligned} \lim_{q\downarrow 1} D_{q,\omega} f(t) &= \Delta_\omega f(t), \\ \lim_{\omega\uparrow 0} D_{q,\omega} f(t) &= D_q f(t), \\ \lim_{q\downarrow 1, \omega\uparrow 0} D_{q,\omega} f(t) &= \frac{d}{dt} f(t), \end{aligned}$$

taking into account that $\downarrow$ and $\uparrow$ mean limits from left and right at finite points respectively. Hahn's operator was applied to construct families of orthogonal polynomials as well as to investigate some approximation problems, see [33, 34, 38]. Another direction of interest is to establish a calculus based on this operator. This was recently studied by M. H. Annaby, A. E. Hamza and K. A. Aldwoah in [5]. They gave a rigorous analysis of the calculus associated with $D_{q,\omega}$, see also [31]. Based on these results, we deduce new results concerning the calculus of this operator and the theory of Hahn difference equations in this thesis. Our results yield those of $q-$difference equations when we take $\theta \to 0$. For more details, see [6] which indicates that the theory of $q-$difference equations may be considered as a special case of Hahn difference equations.

The organization of this thesis is as follows:

► **Chapter One:**

In this Chapter, we introduce the basic concepts and terminology of Hahn's operator calculus. The $q, \omega-$exponential and trigonometric functions with some properties of them are given. At the end of this chapter, we prove some important inequalities on Hahn's calculus like Bernoulli's and Gronwall's Inequalities [22]. Also, we prove Mean Value Theorems associated with Hahn's operator [22].

▶ **Chapter Two:**

In this Chapter, we apply the method of successive approximations to prove the existence and uniqueness theorem of solutions in both local and global cases. The results of this chapter and the final part of Chapter 1 were prepared in one article. It was accepted in "Journal of Advances in difference equations" [22].

▶ **Chapter Three:**

In this Chapter, we study the theory of linear Hahn difference equations of the form

$$a_0(t)D^n_{q,\omega}x(t) + a_1(t)D^{n-1}_{q,\omega}x(t) + ... + a_n(t)x(t) = 0.$$

We introduce its fundamental set of solutions when the coefficients are constant and the Wronskian associated with $D_{q,\omega}$. Hence, we obtain the corresponding Liouville's formula. Also, we derive solutions of the first and second order linear Hahn difference equations with non-constant coefficients.

Finally, we consider the variation of parameter technique and the annihilator method for the nonhomogeneous case. The results of this chapter were prepared in one article. It was accepted in "Journal of Advances in Mathematics" [23].

► **Chapter Four:**

In this Chapter, we establish characterizations of many types of stability, like (uniform, uniform exponential, ψ-) stability of linear Hahn difference equations of the form $D_{q,\omega}x(t) = p(t)x(t) + f(t)$. At the end, we give two illustrative examples. The results of this chapter were prepared in one article. It was accepted in "Journal of Advances in Mathematics" [24].

Chapter 1

Hahn Difference Operator and Associated Jackson-Nörlund Integrals

1.1 Introduction

Hahn introduced the difference operator $D_{q,\omega}$ which extends both Jackson's q-difference operator D_q and the difference operator Δ_ω, see definitions below. Hahn's operator has been considered from computational points of view, in particular in the determination of new families of orthogonal polynomials, see [15, 18, 33]. Calculi, difference equations and special functions based on the q-difference operator as well as the difference operator Δ_ω have been considered extensively, see [4, 17, 30, 37]. Unlike the situation of D_q and Δ_ω, Hahn's difference operator $D_{q,\omega}$ has not received similar attention. In particular, a calculus based on $D_{q,\omega}$ is not rigourously established. For example, the right inverse of $D_{q,\omega}$ is not defined. The purpose of this chapter is to introduce the calculus associated with $D_{q,\omega}$. We state some basic properties from [5]. For instance, a right inverse of $D_{q,\omega}$ is defined in terms of both Jackson q-integral, see [27], which contains the right inverse of D_q and Nörlund sum, cf. [30, 37], which involves the right inverse of Δ_ω. Then, the fundamental theorem of Hahn's calculus is proved. In Section 5, we state some definitions about exponential and trigonometric functions of Hahn's calculus. They satisfy q, ω-difference equations of the first and second order respectively. In Section 6, we prove Gronwall's and Bernoulli's Inequalities. In Section 7, we prove Mean Value Theorems. Sections 2, 3, 4 and 5 were obtained from [5] while those of sections 6 and 7 were published in Journal of Advances in difference equations [22].

1.2 Preliminaries

In [20], Wolfgang Hahn introduced the difference operator $D_{q,\omega}$. It is defined by

$$D_{q,\omega} f(t) = \frac{f(qt + \omega) - f(t)}{(qt + w) - t}, \tag{1.2.1}$$

where $q \in (0, 1)$ and $\omega > 0$ are fixed, see also [21]. Some recent papers have applied this operator to construct families of orthogonal polynomials as well as to investigate some approximation and optimization problems, cf. [12, 20]. This operator unifies and generalizes two well known difference