Faculty of Education
Mathematics Department

Hyperbolically Convex Functions

A Thesis

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(Real Analysis)

Submitted to:

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بِسْمِ اللهِ الرَّحْمنِ الرَّحِيمِ

قَالُواْ سُبْحَانَكَ لاَ عِلْمَ لَنَا إِلاَّ مَا عَلَّمْتَنَا إِنَّكَ أَنتَ الْعَلِيمُ الْحَكِيمُ

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Introduction

The convexity of functions plays a central role in many various fields, such as in economics, mechanics, biological system, optimization, and other areas of applied mathematics. Throughout this thesis, let I be a nonempty, connected, and bounded subset of $\mathbb{R}$. A real valued function $f(x)$ of a single real variable x defined on I is said to be **convex** if for all u, $v \in I$ and $\lambda \in [0, 1]$ one has the inequality:

$$f(\lambda u + (1-\lambda)v) \leq \lambda f(u) + (1-\lambda)f(v). \tag{1}$$

At the beginning of the 20^{th} century, many generalizations of convexity were extensively introduced and investigated in number of ways by numerous authors in the past and present. One way to generalize the definition of a convex function was to relax the convexity condition 1(for a comprehensive review, see the monographs [21]).

As it is well known, the notion of the ordinary convexity can be expressed in terms of linear functions. An important direction for generalization of the classical convexity was to replace linear functions by another family of functions. For instance, Beckenbach and Bing [5, 7], generalized this situation by replacing the linear functions with a family of continuous functions such that for each pair of points $p_1(x_1, y_1)$ and $p_2(x_2, y_2)$ of the plane there exists exactly one member of the family with a graph joining these points.

More precisely, let $\{F(x)\}$ be a family of continuous functions $F(x)$ defined in a real interval I. A function $f : I \to \mathbb{R}$ is said to be sub F-function if for any $u, v \in I$ with $u < v$ there is a unique member of $\{F(x)\}$ satisfying the following conditions:

1. $F(u) = f(u)$ and $F(v) = f(v)$,
2. $f(x) \leq F(x)$ for all $x \in [u, v]$.

The sub F-functions possess various properties analogous to those of classical convex functions [5, 6, 7, 9]. For example, if $f : I \to \mathbb{R}$ is sub F-function, then for any $u, v \in I$, the inequality

$$f(x) \geq F(x)$$

holds out side the interval (u, v).

Of course mathematicians were able before 1937 to generalize the notion of convex functions [32, 33, 39]. In 1908, Phragmén and Lindelöf (see, [22, 32]) dealed with family of trigonometric functions. More precisely, a function $f : I \to \mathbb{R}$ is said to be trigonometrically ρ-convex, if for any arbitrary closed subinterval $[u, v]$ of I such that $0 < \rho(v - u) < \pi$, the graph of $f(x)$ for $x \in [u, v]$ lies nowhere above the unique ρ- trigonometric function, determined by the equation:

$$M(x) = M(x; u, v, f) = A \cos \rho x + B \sin \rho x,$$

where A and B are chosen such that $M(u) = f(u)$, and $M(v) = f(v)$. Equivalently, if for all $x \in [u, v]$

$$f(x) \leq M(x) = \frac{f(u) \sin \rho(v - x) + f(v) \sin \rho(x - u)}{\sin \rho(v - u)}.$$

Full details found in classical books [23, 34, 35] or in the monographs like [27].

In this thesis, we deal just with generalized convexity in the sense of Beckenbach. For particular choices of the two-parameter family

$$F(x) = H(x) = A \cosh px + B \sinh px, \; p \in \mathbb{R} \setminus \{0\},$$

where A and B are chosen such that $H(u) = f(u)$, and $H(v) = f(v)$. which is known as sub H-function.
Actually, this class of functions has three names:

- Sub H-functions in the sense of Beckenbach, see [5] which introduced and studied by Ali in 2016, see [2].
- Hyperbolically convex functions are suggested also by Ali in 2016 as it is analogous to the notion of trigonometrically convex functions which considered by Phragmén and Lindelöf, see [22, 32].
- Hyperbolic p-convex functions considered by Dragomir in 2018, see [13].

On the other side, there are another notion of hyperbolically convex functions in non-Euclidean geometry (relative to hyperbolic geometry which first studied by the Russian mathematician Nikolai Ivanovich Lobachevsky), for more informations see [24, 25].

Currently, we choose hyperbolic p-convex functions as a name for our class as two reasons

- **First**
  the value of p can be used to distinguish between hyperbolic p-convex functions and hyperbolic p-concave functions see Example 2.1.4.
- **Second**
  to avoid ambiguity between hyperbolically convex functions in non-Euclidean geometry and our class.

Finally, in this thesis we study some properties of hyperbolic p-convex functions which analogous to those of classical convex functions. Furthermore, we establish some new integral inequalities of Andersson and Hermite-Hadamard types for hyperbolic p-convex functions. Also, we introduce some applications for special means.

Indeed, there are many reasons for the study of inequalities: practical, theoretical, and aesthetic.

- In many practical investigations, it is necessary to bound one quantity by another. The classical inequalities are very useful for this purpose.

Summary

This thesis is devoted to

1- Discuss one of classes of the generalized convex functions in the sense of Beckenbach which are known as hyperbolic p-convex functions.
2- Study the main characterization of hyperbolic p-convex functions.
3- Extend some properties and integral inequalities (such as: Hermite-Hadamard, Andersson, Ostrowski and Trapezoid, ...) which are known for ordinary convex functions.
4- Introduce some applications for special means.

The thesis consists of five chapters:

Chapter 1

This chapter is an introductory chapter. It contains definitions and basic concepts that are used throughout this thesis. It is regarded as a short survey of the basic needed material.

Chapter 2

The goal of this chapter is to present a short survey of some needed definitions, basic concepts and results of two important vital topics: hyperbolic p-convex functions and supporting functions.

Chapter 3

The purpose of this chapter is to study the standard functional operations of hyperbolic p-convex functions. Furthermore, we prove that the envelope of hyperbolic p-convex functions is hyperbolic p-convex function and introduce a class $BH[a, b]$ of functions representable as the difference of two hyperbolic p-convex functions. The results of this chapter are accepted in Italian Journal of Pure and Applied Mathematics, vol. 43, 2018.

Chapter 4

The main aim of this chapter is to derive three integral inequalities for hyperbolic p-convex functions which are closely connected with Andersson's inequality for ordinary convex functions.

The results of this chapter are

- published in Communications Faculty of Sciences University of Ankara Series A1: Mathematics and Statistics, Vol. 68, 2018.
- presented in the 2nd National Conference for Mathematics and Applications, 2017.

Chapter 5

Finally, in this chapter we prove that the higher powers of $f(x)$ is hyperbolic p-convex function. In addition, we establish some new Hermite-Hadamard type integral inequalities for higher powers of hyperbolic p-convex functions. Also some application for special means are provided as well. The results of this chapter are under submission.

Chapter 1

Preliminaries

The introductory chapter is considered as a background for the material included in the thesis. The purpose of this chapter is to present a short survey of some needed definitions, facts and theories of mathematical analysis. These preliminaries are used in the subsequent chapters but from time to time we supplement them with other results which make the discussion more complete.

1.1 General properties

Definition 1.1.1 *[20] Let $f : [a, b] \to \mathbb{R}$ be a function. Then f is said to be*

- ***increasing*** *on $[a, b]$ if for every $x,\ y \in [a, b]$,*
  $x < y \Rightarrow f(x) \leq f(y)$.
- ***decreasing*** *on $[a, b]$ if for every $x,\ y \in [a, b]$,*
  $x < y \Rightarrow f(x) \geq f(y)$.
- ***monotone*** *if f is either increasing or decreasing on $[a, b]$.*

Definition 1.1.2 *[38] Let f be a real-valued function defined on a set E of real numbers. It is said that f is **continuous at the point** x_0 in E provided that for each $\epsilon > 0$, there is $\delta > 0$ for which*

$$if\ x \in E\ and\ \mid x - x_0 \mid < \delta,\ then\ \mid f(x) - f(x_0) \mid < \epsilon.$$

The function f is said to be ***continuous*** *(on E) provided it is continuous at each point in its domain E.*

Definition 1.1.3 *[36] Suppose $f : [a, b] \to \mathbb{R}$, f is said to satisfy a* ***Lipschitz condition*** *if there exists a constant $K > 0$ such that for every $x, y \in [a, b]$ we have*

$$| f(x) - f(y) | \leq K | x - y | .$$

It is clear that a Lipschitz function is continuous. Indeed, for a number $x \in [a, b]$ and any $\epsilon > 0$, $\delta = \frac{\epsilon}{K}$ responds to the ϵ challenge regarding the criterion for the continuity of f at x. Not all continuous functions are Lipschitz. For example, if $f(x) = \sqrt{x}$ for $0 \leq x \leq 1$, then f is continuous on $[0, 1]$ but is not Lipschitz.

Definition 1.1.4 *[40] A real-valued function f defined on a set E of real numbers is said to be* ***uniformly continuous*** *provided for each $\epsilon > 0$, there is a $\delta > 0$ (δ depends only on ϵ) such that for all x, x_0 in E,*

$$if\ | x - x_0 | < \delta, then\ | f(x) - f(x_0) | < \epsilon.$$

Theorem 1.1.1 *[40] A continuous real-valued function on a closed, bounded set of real numbers is uniformly continuous.*

Definition 1.1.5 *[36] A real valued function f on a closed, bounded interval $[a, b]$ is said to be* ***absolutely continuous*** *on $[a, b]$ provided for each $\epsilon > 0$, there is $\delta > 0$ such that for every finite disjoint collection $\{(a_i, b_i)\}_{i=1}^{n}$ of open intervals in (a, b),*

$$if \sum_{i=1}^{n} [b_i - a_i] < \delta, then \sum_{i=1}^{n} | f(b_i) - f(a_i) | < \epsilon.$$

The criterion for absolute continuity in the case the finite collection of intervals consists of a single interval is the criterion for the uniform continuity of f on $[a, b]$. Thus absolutely continuous functions are continuous. The converse is false, even for increasing functions. For example, Cantor function is continuous but not absolutely continuous.

Definition 1.1.6 *[20] Let* $[a, b]$ *be a closed bounded interval of real numbers, a finite set of points*

$$p = \{x_0, x_1, x_2, x_3, ..., x_m\}$$

satisfying the inequality

$$a = x_0 < x_1 < x_2 < ... < x_m = b$$

is called ***a partition*** *of* $[a, b]$*.*

Definition 1.1.7 *[20] Let* $f : [a, b] \to \mathbb{R}$ *be a function and* $\Pi = \{x_0, x_1, x_2, ..., x_n\}$ *a partition of* $[a, b]$*. It is denoted by* $V_\Pi(f) = \Sigma_{k=0}^{n-1}|f(x_{k+1}) - f(x_k)|$ *and set*

$$V_a^b(f) = \sup_{\Pi} V_\Pi(f),$$

where the supremum is taken over all partitions of $[a, b]$*. A function* f *is said to be of bounded variation on* $[a, b]$ *if* $V_a^b(f)$ *is finite. If* f *is of* ***bounded variation*** *on* $[a, b]$ *we write* $f \in V[a, b]$*. The quantity* $V_a^b(f)$ *is called the total variation of* f *over* $[a, b]$*.*

Theorem 1.1.2 *[20] If* $f : [a, b] \to \mathbb{R}$ *satisfies a Lipschitz condition on* $[a, b]$ *with constant* K*, then* $f \in V[a, b]$ *and* $V_a^b(f) \leq K(b - a)$*.*

1.2 Convex functions on the real line

In this section we present the basic definitions and concepts which will be used later.

Definition 1.2.1 *[34]*
A function $f : I \to \mathbb{R}$ *is called* ***convex*** *if*

$$f(\lambda u + (1 - \lambda)v) \leq \lambda f(u) + (1 - \lambda)f(v) \tag{1.1}$$

for all points u *and* v *in* I *and all* $\lambda \in [0, 1]$*. It is called strictly convex if the inequality (1.1) holds strictly whenever* u *and* v *are distinct points*