RINGS EQUIPPED WITH SEMIDERIVATIONS

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To

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To my family

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Introduction

The available literature includes a number of papers concerning the notion of semiderivations on rings. To our knowledge, the interest in semiderivations in associative rings was initiated in its first occurance in 1983, when Bergen [6] introduced this notion into the scope of prime rings. Since then, some authors have brought interest in studying semiderivations in the subsequent years. In the past three decades they have obtained nice results on semiderivations. More succinctly put, there have been certain results dealing with different kinds of algebraic conditions on semiderivations, as well as two results concerning the structure of a semiderivations on prime rings. If R is a ring, then a mapping $f : R \to R$ is called a semiderivation if there exists a function $g : R \to R$ such that the following conditions hold for all $x, y \in R$: $f(x+y) = f(x)+f(y)$; $f(xy) = f(x)g(y)+xf(y) = f(x)y+g(x)f(y)$; and $f(g(x)) = g(f(x))$. By the way, a derivation $d : R \to R$ is an additive mapping, i.e., $d(x + y) = d(x) + d(y)$, satisfying the product rule $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. It is evident that every derivation is a semiderivation, that is, the class of semiderivations covers that of derivations. As we will see, we may find a semiderivation which is not a derivation.

Our intention in this thesis is to divide its contents into two main portions. The first contains all preliminaries and the main known results in the theme of semiderivations, and even some concerning derivations. This will occupy the first and the second chapters. The second portion is devoted to presenting our results that we have got through working in the thesis. Some of the results obtained my constitute specific extensions of some results appearing in the literature. This work occupies the third, fourth and fifth chapters.

The thesis constitutes of an introduction, five chapters, a list of references, and the arabic summary. The **first chapter** is focused on mentioning the main definitions and concepts that will be used in the thesis. It displays the concepts of commuting maps, centralizing maps, strong-commutativity preserving (scp) maps, derivations, Jordan derivations, prime rings, semiprime rings, right orders, dense ideals, essential ideals, the maximal right ring of quotients, the symmetric ring of quotients, the extended centroid, the central closure, centrally closed rings, and $\Gamma-$prime rings. Moreover, its contains some well-known facts on prime and semiprime rings.

The **second chapter** is devoted to displaying some basic results on derivations and semiderivations on prime and semiprime rings. It constitutes of two sections. In the first section we begin by mentioning a classical result of Herstein on derivations in 1978. He has shown that if R is a prime ring admitting a nonzero derivation d such that $[d(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative whenever $char R \neq 2$, and if $char R = 2$, then either R is commutative or is an order in a simple algebra which is 4-dimensional over its center. Bell and Martindale in 1987 have shown that if R is a semiprime ring admitting a derivation d and U a nonzero left ideal of R such that d is nonzero on U and is centralizing on U, then R contains a nonzero central ideal. In 1992 Daif and Bell have proved that a semiprime ring R is commutative if it admits a derivation d for which either $d([x, y]) = [y, x]$ for all $x, y \in R$, or $d([x, y]) = [x, y]$ for all $x, y \in R$. In 1994 Bell and Daif have shown that if a semiprime ring R admits a strong-commutativity preserving derivation on a nonzero right ideal U of R, then $U \subseteq R$, the center of R. In 1998 Daif has generalized the above mentioned result of Herstein in the following way. Let R be a two-torsion free semiprime ring and U a nonzero ideal of R. If R admits a derivation d which is nonzero on U and $[d(x), d(y)] = 0$ for all $x, y \in U$, then R contains a nonzero central ideal. The second section deals with semiderivations on prime rings. We mention here some of the results therein. In 1983, Bergen has proved that a prime ring R admitting a semiderivation f is an order in a simple algebra finite dimensional over its center, whenever R is an algebra over a commutative ring such that $f^n(R)$ is contained in a finitely generated submodule and $f^{2n-1} \neq 0$. Chang, in 1984 has given six results for semiderivations on prime rings, one of them is an extension of the previously mentioned result of Herstein. Among the rest of his results, he has shown that if f_1 and f_2 are semiderivations of a prime ring R of characteristic not 2 associated with epimorphisms g_1, g_2, respectively, and $f_1 f_2(R) \subseteq Z$, the center of R, then R must be commutative. In 1985, Hongan has proved the following result: If $f \neq 0$ is a semiderivation of a prime ring R of characteristic not two associated with an epimorphism g of R, U is a Lie ideal of R, $W = [U, U]$, and $S = [W, W]$, then the following are equivalent: (i) $U \subseteq Z$, (ii) $W \subseteq Z$, (iii) $S \subseteq Z$, and (iv) $f(U) \subseteq Z$. In 1988, Bell and Martindale have enriched the concept of semiderivations theme with a number of results. For example, if R is a prime ring of characteristic not 2, f is a nonzero semiderivation of R associated with an endomorphism g of R, and there exists a nonzero ideal U of R for which $[f(U), f(U)] \subseteq Z$, then R is commutative. In 1990, Brešar, and Chuang simultaneously, have established the following theorem with two different proofs that describes the structure of semiderivations. If f is a semiderivation of a prime ring R associated with the endomorphism $g : R \to R$, then either one of the following two cases holds:

(i) There exists an element λ in the extended centroid of R such that $f(x) = \lambda(x - g(x))$ for all $x \in R$.

(ii) The endomorphism g is an identity map and f is an ordinary derivation.

In 2004, Ashraf and Ur-Rehman have proved three results on semiderivations, we mention one of them. Let f be a semiderivation of a prime ring R of characteristic not 2 associated with an automorphism g of R, and U a nonzero $(\sigma, \tau)-$Lie ideal of R such that $f(U) \subseteq Z$. Then $U \subseteq Z$. In 2006, Firat has shown that if R is a noncommutative prime ring of characteristic not 2, $a \in R$, and f is a semiderivation of R associated with an epimorphism g of R such that the mapping $x \mapsto [af(x), x]$ is commuting on R, then $a = 0$ or $f = 0$. In the setting of $\Gamma-$rings, Dey and Paul, in 2011, have proved the following result. Let f be a semiderivation of a prime $\Gamma-$ring M associated with the endomorphism $g : M \to M$, then either one of the following two cases holds:

(i) There exists an element λ in the extended centroid of M such that $f(x) = \lambda\delta(x - g(x))$ for all $x \in M$, $\delta \in \Gamma$.

(ii) The endomorphism g is an identity map and f is an ordinary derivation.

In 2013, De Filippis, Mamouni and Oukhtite have showed the following result: Let R be a prime ring of characteristic not 2 and J a nonzero Jordan ideal of R. If R admits a nonzero semiderivation f with associated function g such that $f([x, y]) = [x, y]$ for all $x, y \in J$, then R is commutative or $f(x) = x - g(x)$ for all $x \in R$.

The second portion constituting Chapter 3 focus on our new contributing results in the topic of semiderivations. We will start and finish according to our interest during the time of reading and preparing for the thesis. The **third chapter** includes our first results on semiderivations. some results known in the literature for derivations are extended to semiderivations, and other results on prime rings are generalized to semiprime rings. This chapter is divided into three sections. In the first section we give an extension to a result of Chang (Theorem 2.2.2) in Theorem 3.1.2, which says that: If R is a two torsion free semiprime ring and f is a nonzero semiderivation of R associated with an epimorphism g of R such that $[f(R), f(R)] = \{0\}$, then R contains a nonzero central ideal. In the second section we generalize Theorem 2.1.5 of Bell and Daif in Theorem 3.2.1 which states that if R is a semiprime ring admitting a semiderivation f associated with an epimorphism g of R, and if U is a nonzero ideal of R such that f is scp on U as well as $g(U) = U$, then $U \subseteq Z$. While in Corollary 3.2.5 we deduce from Theorem 3.2.4 that if R is a semiprime ring, U a nonzero ideal of R, and R admits a semiderivation f and a derivation d such that $[f(x), d(y)] = [x, y]$ for all $x, y \in U$, then $U \subseteq Z$. In Theorem 3.2.8 we extend Theorem 2.1.4 of Daif and Bell as follows. Let R be a semiprime ring admitting a semiderivation f associated with an epimorphism g of R for which either $xy + f(xy) = yx + f(yx)$

for all $x, y \in R$, or $xy - f(xy) = yx - f(yx)$ for all $x, y \in R$. Then R is commutative. The third section is devoted to extending Lemma 2.2.12 and Lemma 2.2.13 of Bell and Martindale, respectively, in the following way. Let R be a semiprime ring, and f be a semiderivation on R associated with an endomorphism g of R. If there exists a nonzero essential ideal U of R for which $U \cap g(R) = 0$, then there exists $\lambda \in C$ (the extended centroid of R) such that $f(x) = \lambda(x - g(x))$ for all $x \in R$. Besides, let R be a semiprime ring, and $f \neq 0$ be a semiderivation of R associated with an endomorphism g of R. If g is not one-to-one and V is an essential ideal of R contained in $ker g$, then (a) $f(V)$ is a nonzero ideal of R, and (b) there exists $\lambda \in C$ such that $f(x) = \lambda(x - g(x))$ for all $x \in R$.

In the **fourth chapter** we introduce a new notion concerning semiderivations. Since the zero element 0 of a ring is a central element, we present the so-called the semiderivation-like maps (sdl maps) as a notion covering and including the classical definition of semiderivations. A map $f : R \to R$ is called a semiderivation-like map (an sdl map) associated with an epimorphism $g : R \to R$ if:

(1) $f(x + y) - f(x) - f(y) \in Z$;

(2) $f(xy) - f(x)y - g(x)f(y) \in Z$;

(3) $f(xy) - xf(y) - f(x)g(y) \in Z$; and

(4) $f(g(x)) = g(f(x))$, for all $x, y \in R$.

Of course, every semiderivation is an sdl-map. This chapter is divided into two sections. In Section 4.1, we prove Lemma 4.1.2 which show that if R is any ring with no nonzero central ideals, then every sdl map f on R is additive, i.e., $f(x + y) = f(x) + f(y)$ for all $x, y \in R$. Moreover, we prove Theorem 4.1.3 which states that: Let R be a semiprime ring with no nonzero central ideals, and let f be an sdl map of R associated with an epimorphism g of R, then f satisfies $f(xy) = f(x)y + g(x)f(y) = xf(y) + f(x)g(y)$ for all $x, y \in R$. From Lemma 4.1.2 and Theorem 4.1.3 we get Theorem 4.1.4 in which we deduce that in a semiprime ring R with no nonzero central ideals, every an sdl map associated with an epimorphism of R is a semiderivation of R. In Section 4.2, we get as consequences some known results of Bell and Martindale [5] and Chang [12] in corollaries 4.2.3-9

Chapter 5 is customized to discuss the notion of semiderivations in rings with involution. We introduce the so-called $^*-$semiderivations and Jordan $^*-$semiderivations in $^*-$rings as follows. Let R be a $^*-$ring and let $f : R \to R$ be an additive mapping associated with a mapping g of R. f is said to be a $^*-$semiderivation if it satisfies $f(xy) = f(x)y^* + g(x)f(y) = f(x)g(y) + x^*f(y)$, and $f(g(x)) = g(f(x))$, for all $x, y \in R$. While f is called a Jordan $^*-$ semiderivation if $f(x^2) = f(x)x^* + g(x)f(x) = f(x)g(x) + x^*f(x)$, and $f(g(x)) = g(f(x))$, for all $x \in R$. We can naturally define a $^*-$antisemiderivation. This chapter consists of two sections. In Section 5.1 we present our preliminary results using the abbreviations: $(x^y)_1 = f(xy) - f(x)y^* - g(x)f(y)$, $(x^y)_2 = f(xy) - f(x)g(y) - x^*f(y)$, $\widetilde{(x^y)}_1 = f(xy) - y^*f(x) - f(y)g(x)$, and $\widetilde{(x^y)}_2 = f(xy) - g(y)f(x) - f(y)x^*$, for all $x, y \in R$. In Lemmas 5.1.1-6 we prove some preliminary results concerning the above notions which are used in the next section. In Section 5.2 we get our main results. Theorem 5.2.1 states that: Let R be a noncommutative $^*-$prime ring of characteristic not 2. Suppose that f is a Jordan $^*-$semiderivation of R associated with an endomorphism g of R. If $[R, R] \subseteq Ker\, g$, then f is a $^*-$antisemiderivation of R. In Theorem 5.2.2 we prove the following. Let R be a $^*-$ring of characteristic not 2. Suppose f is a Jordan $^*-$semiderivation of R associated with an endomorphism g of R. If R has a commutator which is not a zero divisor and $[R, R] \subseteq Ker\, g$, then f is a $^*-$antisemiderivation of R. We use Lemma 5.1.1 and Lemma 5.1.2 to prove Theorem 5.2.3, which states that: Let R be a commutative $^*-$prime ring of characteristic not 2. Suppose f is a Jordan $^*-$semiderivation of R associated with an endomorphism g of R. Then f is a $^*-$semiderivation of R.

We would like to mention that the contents of Chapter 3 are collected in [31], and it has been accepted. And the contents of Chapters 4 and 5 are collected, respectively, in [16] and [17], wishing to be awarded their opportunities for publication.