Ain Shams University
Faculty of Education
Department of Mathematics

Sub - implicative ideals in KU - algebras and their applications in fuzzy theory

A THESIS

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To

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SUMMARY

Y. Imai and K. Iseki introduced two classes of abstract algebras: BCK- algebras and BCI-algebras [18-22]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI -algebras. In [3, 11, 16, and 17] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH - algebras. They have shown that the class of BCI- algebras is a proper subclass of the class of BCH- algebras. In[**58**] the authors introduced the notion of d-algebras, which is another useful generalization of BCK- algebras, and then they investigated several relations between d-algebras and BCK- algebras as well as some other interesting relations between d-algebras and oriented digraphs. Y.B. Jun, E. H. Roh and H. S. Kim[26] introduce a new notion, called BH-algebras, which is a generalization of BCH/ BCI /BCK-algebras. They also defined the notions of ideals in BH-algebras. Recently J. Neggers and H. S. Kim[61] introduced the notion of B-algebra, and studied some of its properties. In 1983, (Y. Komori) [39] introduced a notion of BCC-algebras, and (W. A. Dudek) [15] redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Y. Komori. In [15], (W. A. Dudek and X. H. Zhang) introduced a notion of BCC-ideals in BCC algebras and described connections between such ideals and congruences. In 2001, (J.Neggers, S.S.Ahn and H.S.Kim) [59] introduced a new notion, called a Q-algebra and generalized some theorems discussed in BCI/BCK-algebras. Prabpayak and Leerawat [62, 63] introduced a new algebraic structure which is called KU-algebra. They gave the concept of homomorphisms of KU-algebras and investigated some related properties. The concept of a fuzzy set, was introduced in [74]. O. Xi [72] applied the concept of fuzzy to BCK-algebras. In [45,47,52], studied the fuzzification of BCK-algebra and BCI-algebra. In 2002, Mostafa Abd-Elnaby and Yousef [55] introduced the notion of fuzzy ideals of KU-algebras and then they investigated several basic properties which are related to fuzzy KU-ideals. They described how to deal with the homomorphic image and inverse image of fuzzy KU-ideals. They have also proved that the Cartesian product of fuzzy KU-ideals in Cartesian product of fuzzy KU-algebras are fuzzy KU-idealsideals. Liu and Meng [23] introduced the notion of sub-implicative ideals in BCI-algebras. Also Jun [43] introduced the notion

of fuzzy sub-implicative ideals of BCI-algebras and obtained some related interesting properties of these concepts.

Neutrosophic set and neutrosophic logic were introduced in1995 by Smarandache as generalizations of fuzzy set and respectively intuitionistic fuzzy logic. In neutrosophic logic, each proposition has a degree of truth (T), a degree of indeterminancy (I) and a degree of falsity (F),where T,I, F are standard or non-standard subsets of]−0, 1+[, see [28 ,29,66,71].Neutrosophic logic has wide applications in science, engineering, Information Technology, law, politics, economics, finance, econometrics, operations research, optimization theory, game theory and simulation etc. Agboola and Davvaz introduced the concept of neutrosophic BCI/BCK-algebras in [1, 2]. Davvaz B. Davvaz, S M. Mostafa and F.Kareem [13] introduce a neutrosophic KU-algebra and KU-ideal and investigate some related properties.Recently H.Wang et.al [71] introduced an instance of neutrosophic set known as single valued neutrosophic set which was motivated from the practical point of view and that can be used in real scientific and engineering applications.

The thesis deals mainly with new algebraic structure is called sub-implicative ideals in KU-algebras, fuzzy sub-implicative (implicative) ideals of KU-algebras , concepts single valued neutrosophic of sub-implicative ideals in KU-algebras and investigate some related properties.

This thesis has been mainly divided into five chapters. The main text of the thesis is in chapters 1, 2, 3 and 4.

Chapter 0

In this chapter we have given an exhaustive of the basic definitions of some algebras and fuzzy sets which are needed in the subsequent chapters and further, the history of the problem.

Chapter 1

In this chapter, the notions of ku- sub implicative/ ***ku***-positive/ ku- sub-commutative ideals in KU-algebras are established. and some of their properties are investigated. Also, the relationships with ku -sub implicative ideals and ku- sub-commutative/ ***ku***-positive implicative are given.

Chapter 2

In this chapter, We consider the fuzzification of sub-implicative (sub-commutative) ideals in KU-algebras, and investigate some related properties. We give conditions for a fuzzy ideal to be a fuzzy sub-implicative(sub-commutative) ideal. We show that any fuzzy sub-implicative (sub-commutative) ideal is a fuzzy ideal, but the converse is not true. Using a level set of a fuzzy set in a KU-algebra, we give a characterization of a fuzzy sub-implicative (sub-commutative) ideal.

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Chapter 3

In this chapter, we consider *ku* - implicative ideal (briefly implicative ideal) in KU-algebras .The notion of fuzzy implicative ideals in KU - algebras are introduced, several appropriate examples are provided and their some properties are investigated. The image and the inverse image of fuzzy implicative ideals in KU - algebras are defined and how the image and the inverse image of fuzzy ***implicative*** ideals in KU - algebras become fuzzy implicative ideals are studied. Moreover, the Cartesian product of fuzzy implicative ideals in Cartesian product of KU – algebras are given.

Chapter 4

In this chapter, We consider the concepts single valued neutrosophic of sub-implicative ideals in KU-algebras, and investigate some related properties. We give conditions for a single valued neutrosophic ideal to be a single valued neutrosophic sub-implicative ideal. We show that any single valued neutrosophic sub-implicative ideal is a single valued neutrosophic ideal, but the converse is not true. Using a level set of a single valued neutrosophic set in a KU-algebra, we give a characterization of single valued neutrosophic sub-implicative ideal.

LIST OF PUBLICATIONS

1- Mostafa S. M., Omar R. A. K, Abd El- Baseer, O. W.
Sub implicative ideals of KU-Algebras, International Journal of Modern Science and Technology Vol. 2, No. 5, 2017. Page 223-227.

2- S. M. Mostafa, O. W. A. Baseer. Fuzzy Implicative deals of KU-Algebras
J. new theory Year: 2018, Number: 22, Pages: 82-91

3- S. M. Mostafa, O. W. A. Baseer. Fuzzy Sub Implicative deals of KU-Algebras
J. new theory Year: **2018**, Number: **23**, Pages: **1-12**

4- O. W. A. Baseer , S. M. Mostafa single valued neutrosophic sub-implicative ideal of KU-Algebras.at J.new theory Year :2018,Number 25 ISSN:2149-1402

In this chapter we collect all the necessary preliminaries which will be useful in our discussions in the main text of the thesis.

§ *(0.1) BCK /BCI /KU -algebras and some related algebraic structures*

In this section, we review some definitions of BCK/ BCI -algebras and some related algebraic structures. The main results in this section are taken from [18 – 22, 49, 50, 51, 68, 69, and 73]

Preliminaries

Definition 0.1.1 : Let X be a set with a binary operation " * " and a constant 0,then $(X,*,0)$ is called a BCI -algebra ,if it satisfies the following axioms:

(BCI -1) $((x*y)*(x*z))*(z*y)=0$

(BCI -2) $(x*(x*y))*y=0$

(BCI -3) $x*x=0$

(BCI -4) $x*y=0 \, and \; y*x=0 \; implies \, x=y$

for all $x,y,z\in X$

If a BCI -algebra X satisfies the identity $0*x=0$, for all $x\in X$, then X is called a BCK algebra. It is known that the class of BCK -algebras is a proper subclass of the class of BCI -algebras. For brevity X is called a BCK - algebra .

A binary relation $\leq$ in X is defined by : $x \leq y$ if and only if $x*y=0$,then $(X,*,0)$ is a BCK - algebra if and only if it satisfies that :

(BCI `$_1$) : $((x*y)*(x*z))\leq x*y$

(BCI `$_2$) : $(x*(x*y))\leq y$

(BCI `$_3$) : $x\leq x$,

(BCI `$_4$) $x\leq y and \; y\leq x implies \, x=y$

(BCI `$_5$) : $0\leq x$

(BCI `$_6$) : $x\leq y \, if \; and \, only \, if \; x*y=0$.

In a BCK - algebra $(X,*,0)$, the following properties are satisfied :

1. $x \le y$ *implies* $z * x \le z * y$
2. $x \le y$ *and* $y \le z$ *imply* $x \le z$
3. $(x * y) * z = (x * z) * y$
4. $(x * y) \le z$ *implies* $x * z \le y$
5. $(x * z) * (y * z) \le x * y$.
6. $x \le y$ *implies* $x * z \le y * z.$
7. $(x * (x * y)) * (y * x) \le x * (x * (y * (y * x)))$
8. $x * y \le x$
9. $x * 0 = x$

Example 0.1.2 : Let $X = \{0,1,2,3,4\}$ in which * is defined by the following table :

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	0
3	3	1	1	0	0
4	4	1	1	1	0

Then X is BCK - algebra .

Theorem 0.1.3 : An algebra $(X,*,0)$ of type (2,0) is a BCK - algebra if and only if it satisfies the following conditions :

(BCI$_1$) : $((x * y) * (x * z)) * (z * y) = 0$,

(a) : $x * (0 * y) = x$ and

(b) (BCI$_4$) : $x * y = 0 = y * x$ implies $x = y$, for all $x, y, z \in X$, For any $x, y \in X$ denote $x \wedge y = y * (y * x)$

Obviously $x \wedge x = x,\ x \wedge 0 = 0 \wedge x = 0$. But in general , $x \wedge y \neq y \wedge x$

Definition 0.1.4 : Let $(X,*,0)$ be a BCK – algebra , and let S be a non – empty subset of X , then S is called a sub - algebra of X , if for all $x, y \in S,\ x * y \in S$, i.e S is closed under the binary operation $*$ of X .

Example 0.1.5 : Let $X = \{0,1,2\}$ in which * is defined by the following table :

*	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

It is clear that $S = \{0,2\}$ is BCK sub-algebra of X .

Theorem0.1.6 :

Suppose that $(X,*,0)$ is a BCK - algebra and let S be a sub - algebra of X then :

(i) $0 \in S$

(ii) $(S,*,0)$ is also a BCK – algebra

(iii) X is a subalgebra of X

(iv) $\{0\}$ is also a subalgebra of X .

§ 0.2 Bounded BCK-algebras

The main results in this section are taken from [18 – 22, 49,50 ,51, 68 , 69,73]

Definition 0.2.1 : If there is an element 1 of BCK – algebra X satisfying $x \le 1$ for all $x \in X$, then the element 1 is called unit of X . A BCK – algebra with unit is called to be bounded .

Example 0.2.2 : Let $X = \{0,1,2,3,4\}$ in which * is defined by the following table :

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	3	3	0	0
4	4	4	4	4	0

Then $(X,*,0)$ is bounded BCK-algebra with unit 4 .

Note : In a bounded BCK – algebra X , we denote $1 * x$ *by* N_x .

Definition0.2.3 : For a bounded BCK – algebra X , if an element x satisfies $NN_x = x$, then x is called an involution .

Theorem 0.2.4 In a bounded BCK – algebra X , we have :

(a) $N_1 = 0$, $N_0 = 1$,

(b) $NN_x \le X$

(c) $N_y * N_x \le y * x$

(d) $y \le x$ *implies* $N_x \le N_y$

(e) $N_{x*y} = N_{y*x}$

(f) $NNN_x = N_x$

Theorem 0.2.5: In a bounded BCK – algebra X , we have $x * N_y = y * N_x$,

for all $x \, and \, y \in S(X)$ where $S(X)$ is the set of all involutions of a bounded BCK-algebra .

Note that : $NN_0 = N_1 = 0$, and $NN_1 = N_0 = 1$, then the elements 0 and 1 are contained in $S(X)$. Hence $S(X)$ is non - empty.

Theorem 0.2.6 : For any bounded BCK - algebra X , we have $S(X)$ is a bounded sub-algebra of X .

Theorem0.2.7: A BCI –algebra X satisfying $x*(y*z) = (x*y)*z$ is a group in which every element is an involution .

Remark : Let X and Y be BC I -algebras. We define $*$on $X \times Y$ by,

$$(x, y) * (u, v) = (x * u, y * v)$$

for every $(x, y), (u, v) \in X \times Y$. Then $(X \times Y, *, (0, 0))$ is a BC I -algebra.

Definition 0.2.8 Let (X; $*$, 0) be a BCK-algebra .Then X is called a positive implicative if it satisfies for all x, y , z ∈ X , $(x*z)*(y*z) = (x*y)*z$.

Example. 0.2.9 Let X = {0, a, b, 1} be defined by the following table :

$*$	**0**	**a**	**b**	**1**
0	**0**	**0**	**0**	**0**
a	**a**	**0**	**a**	**0**
b	**b**	**b**	**0**	**0**
1	**1**	**1**	**1**	**0**

Then (X; $*$, 0) is a positive implicative BCK-algebra .

Proposition 0.2.10 Let (X; $*$, 0) be a BCK-algebra ,then the following conditions are equivalent : for all x, y, z ∈ X,