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Department of Mathematics

Faculty of Education- Ain Shams University

By

Amira Ragab Abd El-Malek Abd El-Rahman

Supervisors

Prof. Dr. Ahmed Younis Ghaly
Professor of applied Mathematics
Department of Mathematics
Faculty of Education
Ain shams University.

Prof. Dr. Hassan Ahmed Zedan
Professor of Pure Mathematics,
Department of Mathematics,
Faculty of Science,
Kafer El-Sheikh University.

Dr. Seham Shapll Tantawy
Assistant Professor of Pure Mathematics,
Department of Mathematics,
Faculty of Education,
Ain shams University.

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SUMMARY

Chapter I

It is an introduction to the optimal system of subalgebra. Also it introduces fractional differential equations (FDE) including some definitions related Riemann-Liouvill, Capito and their properties. This chapter also includes an introduction to the main concepts of the group analysis, the main concepts of the invariant solutions, the theorems and definitions needed for the study of the invariance properties of differential equations.

Bäcklund transformations (BTs) are also introduced in this chapter.

Chapter II

This chapter is concerned with finding the exact solutions by using Bäcklund transformations (BTs) for the following equations:

the stable fractional nonlinear Schrödinger equation, a Perturbed fractional Nonlinear Schrödinger equation in two cases, the time fractional Korteweg-de Vries (KdV) equation, the time fractional generalized Korteweg-de Vries equation and the time fractional Modified Korteweg-de Vries (FmKdV) equation.

Bäcklund transformations help us to get new solutions for the previous equations in terms of well-known solutions. These known solutions are presented as the constant, the simple and the travelling wave solutions.

Chapter III

This chapter discusses the fractional analysis through studying some definitions and properties which are relevant to it as the definition of Riemann-Liouvill, Capito and their aspects. This chapter also includes several examples to find the symmetry of FDE and using them to obtain exact solutions of equations.

This chapter also highlights the nonlocal symmetry and applying Lie group analysis methods to the class of fractional differential equations containing fractional derivatives of a function with respect to another function.

H. A. Zedan, S. S. Tantawy and A. R. Abdel-Malek, "Invariance of the nonlinear generalized NLS equation under the Lie group of scaling transformations", Nonlinear Dynamics Journal, 82:2001–2005; 2015.

Chapter IV

This chapter pays attention to the Q-symmetry for the equations: the space-time fractional diffusion-wave equation, the space-time fractional nonlinear generalized Schrödinger equation (FNLS) and the space-time fractional Drinfeld's Sokolov-Wilson (FDSW) to obtain the Lie point transformation generators. The application of one-parameter group reduces the number of independent variables consequently these mentioned equations are reduced to set of fractional ordinary differential equations (FODEs) which are solved analytically.

In addition to that, we are interested in the conditional Q-symmetry for the systems mentioned above to transformation them into fractional ordinary differential equations by using symmetry properties. Also, we got new solutions to the previously mentioned systems through conservation laws.

H. A. Zedan, S. S. Tantawy and A. R. Abdel-Malek, "Conservation laws for the space-time fractional of classical Drinfeld's Sokolov-Wilson (FDSW) system", Journal of Fractional Calculus and Applications, 10(2):207-215; 2019.

Chapter V

This chapter aims at determining the optimal subalgebras to fractional differential equations. We transformed the equations: the time fractional generalized Burger's equation and the fractional two-dimensional coupled Burger's equations to fractional ordinary differential equations. Solving these fractional ordinary differential equations, we got new variant solutions the studied equations.

H. A. Zedan, S. S. Tantawy and A. R. Abdel-Malek, "Optimal System of Subalgebras for the Time Fractional Generalized Burgers Equation", Asian Research Journal of Mathematics, 11(4):1-23; 2018.

List of systems studied in my thesis:

1- The Stable Fractional Nonlinear Schrödinger Equation:

$$iq_t^{\alpha} + q_x^{2\alpha} + 2\left|q\right|^2 q = 0;$$

where q represents a normalized complex amplitude of the pulse envelope, t is a normalized distance along the fibre and x is a normalized retarded time.

2- The stable fractional nonlinear generalized Schrödinger equation:

$$i\frac{\partial^{\alpha}\bar{u}}{\partial t^{\alpha}} + \beta\frac{\partial^2\bar{u}}{\partial x^2} + V\left(x\right)\bar{u} + \gamma \mid \bar{u} \mid^2 \bar{u} = 0, \qquad t > 0, \quad 0 < \alpha \leq 1;$$

where $V\left(x\right)$ is the trapping potential and β, γ are areal constants.

3- A Perturbed fractional Nonlinear Schrödinger equation

$$iq_t^{\alpha} + \frac{1}{2}q_x^{2\alpha} + \left|q\right|^2 q + i\varepsilon\beta_1\left(q_x^{3\alpha} + 6\left|q\right|^2 q_x^{\alpha}\right) = 0;$$

where q represents a normalized complex amplitude of the pulse envelope, x is a normalized distance along the fibre, t is a normalized retarded time, ε is a small parameter and β_1 is the coefficient of the linear higher order dispersion effect.

4- The time fractional Korteweg-de Vries (FKdV) equation:

$$q_t^{\alpha} + 6qq_x^{\alpha} + q_x^{3\alpha} = 0.$$

5- The time fractional generalized Korteweg-de Vries equation

$$q_t^{\alpha} + 6\beta\left(q+1\right)qq_x^{\alpha} + \beta q_x^{3\alpha} = 0.$$

6- The time fractional Modified Korteweg-de Vries (FmKdV) equation

$$q_t^{\alpha} + 6q^2 q_x^{\alpha} + q_x^{3\alpha} = 0.$$

7- System of linear fractional partial differential equations (FPDEs):

$$\begin{aligned} D^{\alpha}u - v_x + v + u &= 0, \\ D^{\alpha}v - u_x + v + u &= 0, \qquad 0 < \alpha \leq 1; \end{aligned}$$

where α is a real constant, here u and v are functions of independent variables x and t.

8- The nonlinear system of fractional differential equations (FDEs):

$$\begin{aligned} D^{\alpha}u &= u, && 0<\alpha\leq 1,\\ D^{\beta}v &= 2u^2, && 0<\beta\leq 1,\\ D^{\gamma}z &= 3uv, && 0<\gamma\leq 1; \end{aligned}$$

where α, β and γ are areal constants. Here u, v and z are functions of independent variable x.

9- The space-time fractional diffusion-wave equation:

$$u_t^{\alpha} = ku_{xx}, \qquad 0<\alpha\leq 1, k = constant$$

10- The space-time fractional Drinfel'd–Sokolov–Wilson system (FDSW) as follows

$$\begin{aligned} &\frac{\partial^{\alpha}u}{\partial t^{\alpha}} + p\ v\ v_x = 0,\\ &\frac{\partial^{\alpha}v}{\partial t^{\alpha}} + q\ v_{xxx} + r\ u\ v_x + s\ v\ u_x = 0; \end{aligned}$$

where p, q, r, s are some nonzero parameters.

11- The time fractional generalized Burger's equation:

$$u_t^{\alpha} = u_{xx}^{\beta} + Au^c u_x^{\beta}, \qquad 0<\alpha,\beta\leq 1, c>0.$$

12- The fractional two-dimensional coupled Burger's equations:

$$\begin{aligned} D_t^{\alpha}u + uD_x^{\beta}u + vD_y^{\gamma}u &= \frac{1}{R}\left[D_x^{\beta}D_x^{\beta}u + D_y^{\gamma}D_y^{\gamma}u\right],\\ D_t^{\alpha}v + uD_x^{\beta}v + vD_y^{\gamma}v &= \frac{1}{R}\left[D_x^{\beta}D_x^{\beta}v + D_y^{\gamma}D_y^{\gamma}v\right], \quad 0<\alpha,\beta,\gamma\leq 1; \end{aligned}$$

where u and v are the velocity components to be determined, and R is the Reynolds number.

CHAPTER I

HISTORICAL INTRODUCTION TO FRACTIONAL DIFFERENTIAL EQUATIONS AND LIE GROUP

1.1 Introduction

Fractional calculus is a mathematical branch investigating the properties of derivatives and integrals of non-integer orders (called fractional derivatives and integrals, briefly differintegrals). In particular, this discipline which involves the notion and methods of solving differential equations involving fractional derivatives of the unknown function (called fractional differential equations). The theory of fractional calculus includes even complex orders of differintegrals and left and right differintegrals (analogously to left and right derivatives) [64].

The fact, that the differintegrals an operator which includes both integer-order derivatives and integrals as special cases is the reason why in present fractional calculus becomes very popular and many applications arise. The fractional integral may be used e.g. for better describing the cumulation of some quantity, when the order of integration is unknown, it can be determined as a parameter of a regression model as Podlubny presents in [58]. Analogously the fractional derivative is sometimes used for describing damping.

Fractional calculus is three centuries old as the conventional calculus, but it is not very popular among science and engineering community. The beauty of this subject is that fractional derivatives (and integrals) are not a local (or point) property (or quantity). Thereby this considers the history and non-local distributed effects. In other words, perhaps this subject translates the reality of nature better! Therefore to make this subject available as popular subject to science and engineering community, it adds another dimension to understand or describe basic nature in a better way. Perhaps fractional calculus is what nature understands, and to talk with nature in this language is therefore efficient. For past three centuries, this subject was with mathematicians, and only in last few years, this was pulled to several (applied) fields of engineering, science and economics.

Other applications occur in the following fields: fluid flow, viscoelasticity, control theory of dynamic systems, diffusive transport akin to diffusion, electrical networks, probability and statistics, dynamically processes in self-similar and porous structures, electrochemistry of corrosion, optics and signal processing, rheology etc.

1.2 History of fractional calculus

The concept of Fractional Calculus (FC) is not new and it was as old as calculus itself [19]. The history of FC dates back to more than 300 years ago. In 1695, Leibniz asked a question to de l'Hospital by exchanging the letter: "Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?". That revolutionary question aroused the curiosity of de l'Hospital and on 30^{th} September 1695 he replied to Leibniz with another question: "What if the order will be $\frac{1}{2}$?" Leibniz replied "Thus it follows will be equal to $x^2\sqrt{dx : x}$ an apparent paradox, from which one day useful consequences will be drawn.". Nowadays many scientists consider 30^{th} September 1695 the exact birthday of FC and Gottfried Wilhelm Von Leibniz as the father of FC [19].

Following this nonconventional discussion, in 1730 Leonhard Euler mentioned FC when he studied the interpolation between integer orders of a derivative. Consequently, in 1772, Lagrange developed the exponents for differential operators of integer order (Lagrange, 1849):B. Ross [19]

$$\frac{d^m}{dx^m}\frac{d^n}{dx^n}y = \frac{d^{m+n}}{dx^{m+n}}y. \tag{1.1}$$

This result can be expanded to arbitrary order.

However, the earliest systematic studies of FC were done at the beginning of the 19^{th} century. In 1812, Laplace defined a fractional derivative (FD) for functions by means of an integral and it was first documented in a text in 1819. Starting with $y = x^m$, where m is a positive integer, Lacroix developed the nth derivative

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!}x^{m-n}, \quad m \geq n. \tag{1.2}$$