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hossam maghraby

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hossam maghraby

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HODA EL SHERBINY IBRAHIM EL SHERBINY

Supervisors

Prof. Dr. Peter Schneider
Department of Mathematics
Westfaelische Wilhelms–
Universitaet Münster
Germany

Prof. Dr. Nashat Faried
Department of Mathematics
Faculty of Science
Ain Shams University
Cairo, Egypt

Dr. Ismail M. Idris
Department of Mathematics
Faculty of Science
Ain Shams University
Cairo, Egypt

To

Department of Mathematics
Faculty of Science
Ain Shams University
Cairo, Egypt
2001

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Summary

Summary

In 1949, André Weil, [25], declared his now well known conjectures concerning the number of solutions of polynomial equations over finite fields. His conjectures suggested a deep connection between the arithmetic of algebraic varieties defined over finite fields and the topology of algebraic varieties defined over the complex numbers. Weil explained that if one had a suitable good cohomology theory for abstract varieties, analogous to the ordinary cohomology of varieties defined over $\mathbb{C}$, one could deduce his conjectures from various standard properties of the cohomology theory. This observation was one of the principal motivations for the introduction of various cohomology theories into abstract algebraic geometry. In 1955, [21], Serre introduced the first cohomology theory into abstract algebraic geometry using coherent sheaves on algebraic varieties with respect to the Zariski topology. This was an algebraic analogue of the notion of coherent sheaves in analytic geometry. Some years later, Grothendieck, [4], inspired by some of the Serre's ideas. He could obtain a good theory by considering the variety together with all its unramified covers. This was the beginning of his theory of étale topology, developed jointly with M. Artin, which he used to define the p-adic cohomology. The crystalline cohomology of Grothendieck, [8], and Berthelot, [1], gives another similar cohomological interpretation of the Weil conjectures. In, [8], Grothendieck established a relation between étale cohomology and de Rham cohomology. We recall very briefly his ideas as follows:

Let p be a prime number and $\mathfrak{X}$ be a smooth projective scheme over the discrete valuation ring $\mathbb{Z}_p$ with a generic fibre $X = \mathfrak{X} \times_{\mathbb{Z}_p} \bar{\mathbb{Q}}_p$ where $\bar{\mathbb{Q}}_p$ is the algebraic closure of $\mathbb{Q}_p$ and a special fibre $Y = \mathfrak{X} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. The $\mathbb{Q}_p$-adic étale cohomology associated to the generic fibre X is denoted by $V_X = H^*(X_{\text{ét}}, \mathbb{Q}_p)$; and the crystaline cohomology associated to the special fibre Y is denoted by $D_Y = H^*(X_{\text{zar}}, \Omega^\bullet_{X/\mathbb{Q}_p})$; where $\Omega^\bullet_{X/\mathbb{Q}_p}$ is the complex of differential forms on X.

Both of the cohomology groups V and D are $\mathbb{Q}_p$ vector spaces of the same finite dimension but equipped with different structures. The

$\mathbb{Q}_p$ vector space V is a representation of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ while the $\mathbb{Q}_p$ vector space D is endowed with Hodg filtration together with a frobenius structure $\varphi\colon D \longrightarrow D$.

Grothendieck raised the problem of finding an explicit recipe for passing between V and D. This is his problem of the "mysterious functor"!. In 1987 Fontaine and Messing , [18], invented a "fantasy" tool to prove the comparison isomorphism between crystalline and p-adic étale cohomology or the isomorphism between D and V. This fantasy tool of Fontaine and Messing is the syntomic cohomology or more precisely the cohomology of sheaves on the syntomic site of a scheme. This is the main topic of the present work.

The thesis consists of four chapters.

The first chapter includes the necessary tools from homological Algebra that will be used in the sequel. It introduces also one of the main objects in algebraic geometry, namely ,schemes.

The first section covers the general concepts and definitions relevant to derived functors, which are necessary to introduce the next section. The second section gives a clear and careful development of the basic facts of spectral sequences, which will be used in Chapter IV. The third section provides a bridge between commutative algebra and homological algebra. It gives the necessary and sufficient condition for a sequence of elements of a commutative ring to be regular in terms of the homology groups of the associated Koszul complex of this sequence of elements. In the fourth section we follow the elegant exposition of Hartshorne of foundations of the definition of scheme, morphisms of schemes and mentioning some properties of schemes.

The second chapter is essential for the following chapters. We introduce different classes of morphisms of schemes. All of which are stable under composition and base change.

The first section gives a clear exposition of the definitions and basic facts concerning quasi-coherent and coherent sheaves. This is another link between commutative algebra and abstract algebraic geometry, as we have a faithful functor from the category of A-modules