Trigonometrically Convex Functions

A Thesis

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(Real Analysis)

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بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

"وقل ربي زدني علما"

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In the name of Allah most graceful most merciful
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Contents

List of Abbreviations and Symbols

We list up basic notations, abbreviations and symbols to be used in the sequel:

1. $\mathbb{R}$: is the field of real numbers.

2. $\mathbb{R}^+$: is the set of all positive real numbers.

3. $\mathbb{Z}$: is the set of integers.

4. $\mathbb{N} = \{1, 2, 3, ...\}$: is the set of natural numbers.

5. $\emptyset$: empty set.

6. I^o: the interior of set I.

7. $f|_K$: restriction of f to K.

8. $f'_-(x)$, $f'_+(x)$: left and right hand derivatives of $f : I \to \mathbb{R}$ at x.

9. $\partial f(x)$: subdifferential of f at x.

10. f^*: conjugate function.

Introduction

The convexity of functions play a central role in many various fields, such as in economics, mechanics, biological system, optimization, and other areas of applied mathematics. Throughout this thesis, let I be a nonempty, connected, and bounded subset of $\mathbb{R}$.

An important mathematical problem is to investigate how functions behave under the action of means. The best-known case is that of midpoint convex (or Jensen convex) functions, which deal with the arithmetic mean. As it is well know, the recognition of convex functions as a class of functions to be studied is generally traced to Jensen. They are precisely the functions $f : I \to \mathbb{R}$ such that

$$f(\frac{u+v}{2}) \leq \frac{f(u) + f(v)}{2},$$

for all u and v in I. Mid point convexity means convexity which can be defined as follows: A real valued function $f(x)$ of a single real variable x defined on I is said to be **convex** if for all u, $v \in I$ and $\lambda \in [0, 1]$ one has the inequality:

$$f(\lambda u + (1 - \lambda)v) \leq \lambda f(u) + (1 - \lambda)f(v). \tag{1}$$

At the beginning of the 20th century, many generalizations of convexity were extensively introduced and investigated in number of ways by numerous authors in the past and present. One way to generalize the definition of a convex function was to relax the convexity condition (1) (for a comprehensive review, see the monographs [31]).

As it is known, the notion of the ordinary convexity can be expressed in terms of linear functions. An important direction for generalization of the classical convexity was to replace linear functions by another family of functions. For instance, Beckenbach and Bing [9, 10, 11], generalized this situation by replacing the linear functions with a family of continuous functions such that for each pair of points $p_1 : (u_1, v_1)$ and $p_2 : (u_2, v_2)$, of the plane there exists exactly one member of the family with a graph joining these points.

More precisely, let $\{F(x)\}$ be a family of continuous functions $F(x)$ defined in a real interval I. A function $f : I \to \mathbb{R}$ is said to be sub F-function if for any $u, v \in I$ with $u < v$ there is a unique member of $\{F(x)\}$ satisfying the following conditions:

1. $F(u) = f(u)$ and $F(v) = f(v)$,

2. $f(x) \leq F(x)$ for all $x \in [u, v]$.

The sub F-functions possess various properties analogous to those of classical convex functions [12, 16, 17, 25, 42, 53]. For example, if $f : I \to \mathbb{R}$ is sub F-function, then for any $u, v \in I$, the inequality

$$f(x) \geq F(x)$$

holds out side the interval (u, v).

Of course mathematicians were able before 1937 to generalize the notion of convex functions [43, 45, 52].

In 1908, phragmén and Lindelöf [43] (see, e.g. [34]) showed that if $f(z)$ is an entire function of order $0 < \rho < \infty$, then its indicator which is defined as:

$$h_f(\theta) = \limsup_{r \to \infty} \frac{\log |f(re^{i\theta})|}{r^\rho}, \qquad (2)$$

where $\alpha < \theta < \beta$.

For the function

$$f(z) = e^{(A-iB)z^\rho}, \tag{3}$$

holomorphic in an angle $\{z = re^{i\theta} : \alpha < \theta < \beta\}$, such that $\beta - \alpha \le 2\pi$, we have

$$
\begin{aligned}
|f(re^{i\theta})| &= |e^{(A-iB)(\cos\rho\theta + i\sin\rho\theta)r^\rho}| \\
&= e^{(A\cos\rho\theta + B\sin\rho\theta)r^\rho}
\end{aligned} \tag{4}
$$

by substituting in (2), then its indicator is equal to

$$M(\theta) = A\cos\rho\theta + B\sin\rho\theta.$$

Such functions are called sinusoidal or ρ-trigonometric.

If $0 < \theta_2 - \theta_1 < \frac{\pi}{\rho}$, then the sinusoidal function $M(\theta)$ assuming values h_1 and h_2 at the point θ_1 and θ_2 is unique and can be expressed by the formula

$$M(\theta) = \frac{h_1 \sin\rho(\theta_2 - \theta) + h_2 \sin\rho(\theta - \theta_1)}{\sin\rho(\theta_2 - \theta_1)},$$

where $\theta_1 \le \theta \le \theta_2$. This property is called a trigonometrically ρ-convexity. (See for example [33, 34, 35]). If $\rho = 1$ this property is called a trigonometrically convexity. In 1929, Pólya [45] showed that trigonometric convex functions have certain differential properties in common with convex functions, and in 1932 Valiron [52] extended the analysis to functions $f(x)$ dominated by functions of the form:

$$\lambda\phi(x) + \delta\psi(x)$$

where the functions $\phi(x)$ and $\psi(x)$ defined whatever x and admitting continuous first derivatives and to be bounded variation.

In this thesis, we deal just with generalized convexity in the sense of Beckenbach. More actually, a function $f : I \to \mathbb{R}$ is said to be trigonometrically ρ-convex, if for any arbitrary closed subinterval $[u, v]$ of I such that $0 < \rho(v - u) < \pi$, the graph of $f(x)$ for $x \in [u, v]$ lies nowhere above the unique ρ- trigonometric function, determined by the equation:

$$M(x) = M(x; u, v, f) = A\cos\rho x + B\sin\rho x,$$

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where A and B are chosen such that $M(u) = f(u)$, and $M(v) = f(v)$. Equivalently, if for all $x \in [u, v]$

$$f(x) \leq M(x) = \frac{f(u) \sin \rho(v - x) + f(v) \sin \rho(x - u)}{\sin \rho(v - u)}.$$

Trigonometrically ρ-convex functions play an important role in many various aspects in the theory of entire functions of order ρ (see for example [33, 34, 35]) and in the theory of cavitation diagrams for hydrofoils, (see for example [1, 4, 6, 8]). Furthermore, in a problem in geometry [24, 25] and an extremum property [2].

- A hydrofoil is simply a lifting surface, or foil, that operates in water. These are similar to aerofoils used in aeroplanes. As a hydrofoil craft gains speed, the hydrofoils lift the boat's hull out of the water. It decreases drag and allows greater speeds. In [8], they describe basic aspects of the theory of pressure which allows to modify a series of hydrofoils designed by Eppler. These modifications depend on the maximum velocity that is trigonometrically convex function. The properties of trigonometrically convex functions are used to get the best design of the hydrofoil which increase the velocity of the craft. For more details in section 6.1.

- In geometry [24], let C be a closed circular area in the plane, C', its boundary, and K, a closed convex set in C which subtends at every point of C' the same angle α, $0 < \alpha < \pi$. By this is meant that, at each point P of C' the angle between the two extreme supporting half lines to K at P is equal to α. If K is a circular area concentric with C and of radius $\sin \frac{1}{2}\alpha$ times the radius of C, it does subtend the angle α on C'. The question arises then as to whether or not the fact that K subtends a constant angle on C' implies that K is such a circle. It is shown in the following that the answer depends on the nature of the angle α and supporting

function of a convex set. Green gave A necessary and sufficient condition that a function of period 2π be the supporting function of a convex set K is that a function be a sub-sin function (trigonometrically convex function).

- In extremum property. Mohamed S. S. Ali [2] showed that the integration of difference between trigonometrically ρ-convex function and its supporting function has a minimum value at middle of the interval.

Conclusion, in this thesis we study some properties of trigonometrically ρ-convex functions which analogous to those of classical convex functions. Moreover, we introduce a definition of conjugate trigonometrically ρ-convex functions by using Young's inequality which plays an important role in linking the concept of duality between trigonometrically ρ-convex functions. Furthermore, we show that the integration of any increasing function is trigonometrically ρ-convex. In addition, we established some new integral inequalities of Pólya, Steffensen, Young, Hermite-Hadamard and Cauchy-Schwarz types for trigonometrically ρ-convex functions. Also, we study some properties of the multiplication of two trigonometrically ρ-convex functions, and prove the non negative convex function is trigonometrically ρ-convex functions. Finally, we give applications of trigonometrically convex functions in different fields.

Summary

This thesis is devoted to

1. Discuss one of classes of the generalized convex functions in the sense of Beckenbach which are known as trigonometrically ρ-convex functions.

2. Study the main characterization of trigonometrically ρ-convex functions.

3. Extend some properties and integral inequalities such as: Young, Pólya, Steffensen, Hermite-Hadamard, Cauchy-Schwarz.

4. Introduce applications of trigonometrically convex functions.

The thesis consists of six chapters:

Chapter 1

This chapter is an introductory chapter. It contains definitions and basic concepts that are used throughout this thesis. It is regarded as a short survey of the basic needed material.

Chapter 2

The goal of this chapter is to present a short survey of some needed definitions, basic concepts and results of these two important vital topics: trigonometrically ρ-convex functions and supporting functions. Also, some integral inequalities for Hermite-Hadamard and for higher powers of trigonometrically ρ-convex functions are showed.

Chapter 3

The purpose of this chapter is to introduce a definition of conjugate trigonometrically ρ-convex functions by using Young's inequality which plays an important role in linking the concept of duality between trigonometrically ρ-convex functions, rather the definition given by Fenchel. Furthermore, we show that the integration of any increasing functions are trigonometrically ρ-convex functions.

Some results of this chapter are:

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- Presented in the 2nd National Conference for Mathematics and Applications, Cairo, Egypt, 2017.