Geometrical Local Structures of Banach Spaces and Dvoretzky Theorem

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Summary

This study is mainly devoted to the local theory of Banach spaces. In this theory, one obtains information regarding an infinite dimensional Banach space from its local structure - the collection of all its finite dimensional subspaces or quotients. The first major result of the local theory is Dvoretzky's Theorem 1960. In his original approach, Dvoretzky used the Haar measure on the Grassmann manifold of all k-dimensional subspaces of $\mathbb{R}^n$. The objective of the first three chapters is to present self-contained proofs of two fundamental results Dvorezky-Rogers theorem and Dvoretzky theorem on almost spherical (rather ellipsoidal) sections of convex bodies. Then we display consequences of Dvoretzky theorem in the fourth chapter, and one of its consequences is a stronger version of Dvoretzky-Rogers theorem. Finally, we get new results in this direction in the last chapter.

So, this M. Sc. thesis is organized as follows:

In chapter ♯ 1, we study Haar measure, the quotient topology and how the subspace topology on the Stiefel manifold gives a quotient topology on the Grassmann manifold. Also, we mention preliminaries on Banach spaces and linear operators and study Banach-Mazur distance.

The most important concept studied was the correspondence between Banach spaces and symmetric convex bodies. This allows one to use arguments from convex geometry in functional analysis and vice versa. Furthermore, we studied the geometric notion of Banach-Mazur

distance and showed how the analytic and geometrical notions are connected through the correspondence between normed spaces of dim n and symmetric convex bodies in R^n.

In chapter ♯ 2, we study one example of duality which is polarity. Also, we study support functions and distance functions and derive some of their basic properties and establish the dual relationship between them .

In chapter ♯ 3, we are exposed to geometry of Banach spaces. we give an idea about absolute and unconditional convergence of series in linear topological space. We study Dvoretzky-Rogers theorem and Dvoretzky theorem on almost spherical (rather ellipsoidal) sections of convex bodies. The objective of chapter 1 and 2 is to present self-contained proofs for them in this chapter.

In chapter ♯ 4, we conclude consequences of Dvoretzky theorem on almost spherical (rather ellipsoidal) sections of convex bodies. We give a dual version of Dvoretzky theorem. One of its important consequences is isometric characterization of Hilbert spaces. Furthermore, it give a stronger version of Dvoretzky-Rogers theorem

In chapter ♯ 5, This motivated us to give rigrous formulas which facilitate calculating the center and the radius of the smallest ball containing a set K in a linear normed space, and the center and the radius of the largest ball contained in it provided that K has a nonempty boundary set with respect to the flat space generated by it. Also, we give a formula to calculate the asphericity for each set has a nonempty boundary set with respect to the flat space generated by it. We proof that this definition is equivalent to the one given by Dvoretzky. This allowed us to get lower and upper estimations for the asphericity of infinite and finite cross product of these sets in certain spaces, respectively.

Introduction

This study is mainly devoted to the local theory of Banach spaces. In this theory, one obtains information regarding an infinite dimensional Banach space from its local structure - the collection of all its finite dimensional subspaces or quotients. The first major result of the local theory is Dvoretzky's theorem. This theorem gave an affirmative answer to the conjecture raised by A. Grothendieck [26] in 1956 "Pour n and ϵ donn*é*s, tout espace de Banach E de dimension assez grande contient un sous-espace isomorphe *à* ϵ près *à* l'espace de Hilbert de dimension n". Dvoretzky proved that for any infinite dimensional Banach space X and for any $\epsilon > 0$ and natural number n there exists a subspace L of X with dim L=n s.t. $1 \leq d(L, \mathbb{R}^n) \leq 1+\epsilon$ where d is the Banch-Mazur distance. In fact, he proved that every symmetric convex body (compact convex set with non-empty interior) of sufficiently high dimension n admits an almost-spherical (or rather ellipsoidal) k-dimensional section i.e., given ϵ, 0< ϵ <1, and a positive integer k. Then there exists an integer N=N(k;ϵ) such that if C is any symmetric convex body in $\mathbb{R}^n$(real Euclidean n-space), n$\geq$ N, there is a subspace $\mathbb{R}^k$ for which

$$\alpha(C \cap \mathbb{R}^k) < \epsilon. \tag{1}$$

i.e. there is a subspace $\mathbb{R}^k$ and a positive number r so that $B_{r(1-\epsilon)} \subset C \cap \mathbb{R}^k \subset B_r$, where $B_s = \{x \in \mathbb{R}^k : |x| \leq s\}$. The one-to-one correspondence between n-dimensional normed spaces and n-dimensional symmetric convex bodies easily shows the result. Dvoretzky [17] noted that by duality, every symmetric convex body of sufficiently high dimension n admits an almost-spherical (or rather ellipsoidal) k-dimensional projection. Straus [46] in 1963 showed that there is actually a subspace for which both the section and the projection are almost spherical. Due to lack of studying modern Banach space theory in egypt, this study has been focused on selected topics of this theory.

This motivated us to give rigrous formulas which facilitate calculating the center and the radius of the smallest ball containing a set K in a linear normed space, and the center and the radius of the largest ball contained in it provided that K has a nonempty boundary set with respect to the flat space generated by it. Also, we use a formula to calculate the asphericity for each set has a nonempty boundary set with respect to the flat space generated by it. This allowed us to get lower and upper estimations for the asphericity of infinite and finite cross product of these sets in certain spaces, respectively.

Chapter 1

Functional analysis and convex geometry

1.1 Haar measure

Definition 1.1.1 *[28] A σ-ring is a nonempty class S of sets such that*
(i) if $E \in S$ and $F \in S$, then $E - F \in S$, and
(ii) if $E_i \in S$, $i = 1, 2, ...,$ then $\cup_{i=1}^{\infty} E_i \in S$.

Definition 1.1.2 *[28] A non-empty set G together with a binary operation $*$ is said to form a group if $(G, *)$ satisfies the following properties:*
(i) For all $a, b \in G \to a * b \in G$,
(ii) $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$,
(iii) There exists an element $e \in G$ such that $a * e = e * a$ for all $a \in G$, then e is called the identity element of G.
(iv) To every $a \in G$, there exists an element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$, where e is the identity element of G and a^{-1} is called the inverse element of a.

Definition 1.1.3 *[40] Let X and Y be topological spaces. A function $f : X \to Y$ is said to be continuous if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X. Recall that $f^{-1}(V)$ is the set of all* *points x of X for which $f(x) \in V$; it is empty if V does not intersect the image set $f(X)$ of f.*

Continuity of a function depends not only upon the function f itself, but also on the topologies specified for its domain and range. Let us note that if the topology of the range space Y is given by a basis B, then to prove continuity of f it suffices to show that the inverse image of every basis element is open. In fact, the arbitrary open set V of Y can be written as a union of basis elements $V = \bigcup_{\alpha \in J} B_\alpha$. Then $f^{-1}(V) = \bigcup_{\alpha \in J} f^{-1}(B_\alpha)$, so that $f^{-1}(V)$ is open if each set $f^{-1}(B_\alpha)$ is open.

Definition 1.1.4 *[11] A topological group is a set G that has the structure of a group (say with group operation $(x, y) \mapsto xy$) and of a topological space, and is such that the operations $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous.*

Note that $(x, y) \mapsto xy$ is a function from the product space $G \times G$ to G, and that we are requiring that it be continuous with respect to the product topology on $G \times G$; thus xy must be "jointly continuous" in x and y, and not merely continuous in x with y held fixed and continuous in y with x held fixed.

Definition 1.1.5 *[40] Let X and Y be topological spaces. The product topology on $X \times Y$ is the topology having as basis the collection B of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.*

Example 1.1.6 *[11] Let G be $\mathbb{R}$ with addition as the group operation and with the topology for which the open sets are those that either are empty or have a countable complement then $x \mapsto -x$ is continuous, $(x, y) \mapsto x + y$ is continuous if either one of the entries is held fixed i.e. is continuous in x when y is held fixed and continuous in y when x is held fixed but that it is not jointly continuous i.e. $(x, y) \mapsto x + y$*

is not continuous.
Thus G is not a topological group.

Proof Suppose that $x_0 \in G$ is fixed. Let U be open in $\mathbb{R}$. Then, U:=$\mathbb{R}$-$\{x_n\}_{n=1}^{\infty}$ for some countable collection $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$. The complete preimage of U is

$$\begin{aligned}\{(x_0,y)\in\{x_0\}\times\mathbb{R} : x_0+y\in U\} &= \{(x_0,y)\in\{x_0\}\times\mathbb{R} : x_0+y\neq x_n, n\in N\}\\ &= \{(x_0,y)\in\{x_0\}\times\mathbb{R} : y\neq x_n-x_0, n\in N\}\\ &= \{x_0\}\times(\mathbb{R}-\{x_n-x_0\}_{n=1}^{\infty}).\end{aligned}$$

This is a relatively open subset of $\{x_0\} \times \mathbb{R}$. Hence, the map $[(x_0,y) \longmapsto x_0+y]$: $\{ x_0 \} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous. Using an analogous argument, the map $[(x,y_0) \longmapsto x+y_0]$: $\mathbb{R} \times \{y_0\} \longrightarrow \mathbb{R}$ (for some y_0 fixed in $\mathbb{R}$) is continuous.

Now, consider the set $\{0\} \subset \mathbb{R}$. This is closed in our topology since $\mathbb{R}$-$\{0\}$ has countable complement. The complete preimage of $\{0\}$ is

$$\{(x,y) : x+y=0\} = \{(x,y) : y=-x\}$$

under the map $[(x,y) \longrightarrow x+y]$. This is the line $y=-x$ in $\mathbb{R}^2$. If the map $[(x,y) \longrightarrow x+y]$ is continuous, then the line $y=-x$ must be closed in $\mathbb{R}\times\mathbb{R}$ with the product topology. So, consider any point $(p,q) \in \mathbb{R}^2$ so that $p \neq -q$. If the line $y=-x$ is closed, we must be able to find a basis open neighborhood $U \times V$ so that $(p,q) \in U \times V$ and $U \times V \bigcap \{ y=-x \} = \emptyset$. U and V have the form

$$U := \mathbb{R}-\{u_n\}_{n=1}^{\infty} \qquad V := \mathbb{R}-\{v_n\}_{n=1}^{\infty}$$

for $p \notin \{u_n\}_{n=1}^{\infty}$ and $q \notin \{v_n\}_{n=1}^{\infty}$. Then,

$$\begin{aligned}U\times V &= \{(x,y) : x\in U, y\in V\}\\ &= \{(x,y) : x\notin\{u_n\}_{n=1}^{\infty}, y\notin\{v_n\}_{n=1}^{\infty}\}\\ &= \mathbb{R}\times\mathbb{R}-\{(u,v) : u\in\{u_n\}_{n=1}^{\infty}, v\in\{v_n\}_{n=1}^{\infty}\}.\end{aligned}$$

That is, $U \times V$ is $\mathbb{R}^2$ minus a countable collection of points(Cross product of countable sets is countable). But, $\{y = -x\}$ is uncountable. Thus, $U \times V \bigcap \{ y = -x \} \neq \emptyset$. Thus, $\{y = -x\}$ is not closed and the map $[(x, y) \longrightarrow x + y]$ is not continuous.

Example 1.1.7 *[11] Let G be $\mathbb{R}$, with addition as the group operation and with the weakest topology that makes each interval of the form $(a, b]$ open. Then $(x, y) \mapsto x + y$ is continuous, but that $x \mapsto -x$ is not continuous. Thus G is not a topological group.*

Proof First, we show that the topology generated by the half-open intervals $(a, b]$ is the weakest topology with these intervals open. It is well-known that these half-open intervals form a basis for the topology known as the upper-limit topology. This is the weakest such topology since any topology containing the intervals $(a, b]$ must contain arbitrary unions of the intervals $(a, b]$. Thus, it contains each open set in the upper-limit topology. That is, the upper-limit topology which is generated by the half-open intervals $(a, b]$ is the weakest such topology.

Now, consider the open set $(0, 1]$. Then, the complete preimage of this set is $[-1, 0)$ under the map $[x \longrightarrow -x]$. We will show that $[-1, 0)$ is not open by showing that there is no basis element $(a, b]$ so that $-1 \in (a, b] \subseteq [-1, 0)$. If $-1 \in (a, b]$, then $a < -1 \leq b$. But, then, there is some $x \in (a, -1)$ so that $a < x < -1 \leq b$. That is, $x \in (a, b]$, but $x \notin [-1, 0)$. Thus, $[-1, 0)$ is not open in our topology and the map $[x \longrightarrow -x]$ is not continuous.

Proposition 1.1.8 *[11] Suppose that* **G** *is a group and a topological space then* **G** *is a topological group if and only if the map (x,y) $\mapsto xy^{-1}$ from* **G** $\times$ **G** *to* **G** *is continuous.*

Proof "$\Rightarrow$" Notice that the map $[(x, y) \mapsto (x, y^{-1})]$ is continuous. To prove this, let $U \times V$ be a basis open set in $G \times G$. Then,

$$\begin{aligned} U \times V^{-1} &= \{(x, y) : x \in U, y \in V^{-1}\} \\ &= \{(x, y) : x \in U, y^{-1} \in V\} \\ &= \{(x, y) : (x, y^{-1}) \in U \times V\} \end{aligned}$$

is the complete preimage of $U \times V$ under the map $[(x, y) \mapsto (x, y^{-1})]$. Since U is open and V^{-1} is open as the complete preimage of V under the assumed continuous map $[x \mapsto x^{-1}]$, $U \times V^{-1}$ is open. Since preimages conserve unions and any any open set in $G \times G$ is the union of basis sets $U \times V$, $[(x, y) \mapsto (x, y^{-1})]$ is continuous. Finally, since the map $[(x, y) \mapsto xy^{-1}]$ is the composition of the continuous map $[(x, y) \mapsto (x, y^{-1})]$ with the assumed continuous map $[(x, y) \mapsto xy)]$, $[(x, y) \mapsto xy^{-1}]$ is continuous. That is,

$$(x, y) \mapsto (x, y^{-1}) \mapsto xy^{-1}$$

is continuous.

"$\Leftarrow$" Consider the map $[x \mapsto x^{-1}]$. This is just a composition of the homeomorphism $[x \mapsto (e, x)] : G \to \{e\} \times G$ (where e is the group identity of G) and the assumed continuous map $[(x, y) \mapsto xy^{-1}]$. That is,

$$x \mapsto (e, x) \mapsto ex^{-1} = x^{-1}$$

is continuous since $ex = x$ for each $x \in G$. Now, just as in the previous direction, $[(x, y) \mapsto (x, y^{-1})]$ is continuous. Thus, $[(x, y) \mapsto xy]$ is just the composition of the continuous map $[(x, y) \mapsto (x, y^{-1})]$ with the assumed continuous map $[(x, y) \mapsto xy^{-1}]$. That is,

$$(x, y) \mapsto (x, y^{-1}) \mapsto x(y^{-1})^{-1} = xy$$

is continuous since $(y^{-1})^{-1} = y$.

Definition 1.1.9 *[28] A space X is locally compact if every point of X has a neighborhood whose closure is compact.*