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Oscillation Criteria for Impulsive Dynamic Equations on Time Scales

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Contents

Summary

The study of dynamic equation on a time scales goes back to its founder Stefan Hilger [16]. He aims to unify, extend and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time scale calculus. A time scale $\mathbb{T}$ is a nonempty closed subset of real numbers. When the time scale is the set of real numbers, the general results yield the results of ordinary differential equations. On the other hand, when the time scale is the set of integers, the same general results yield the results of difference equations. The new theory of the so-called "dynamic equation" is not only unify the theories of differential equations and difference equations, but also extends the classical cases to the so-called q-difference equations (when $\mathbb{T} = q^{\mathbb{N}_0} := \{q^t : t \in \mathbb{N}_0,\ q > 1\}$ or $\mathbb{T} = q^{\overline{\mathbb{Z}}} = q^{\mathbb{Z}} \cup \{0\}$) which have important applications in quantum theory (see Kac and Chueng [23]).

Impulsive differential equations, that is, differential equations involving an impulse effect, appear as a natural description of observed evolution phenomena of several real-world problems. The first investigation of the oscillation theory of impulsive differential equations was published in 1989 (Gopalsamy and Zhang [21]). In recent years, there has been increasing interest in studying oscillation (nonoscillation) of the solutions of impulsive dynamic equations on time scales. So, we choose the title of the thesis " Oscillation Criteria for Impulsive Dynamic Equations on Time Scales". We aim to use the generalized Riccati transformation and the impulsive inequality to establish some new oscillation criteria for the second-order nonlinear impulsive dynamic equation on a time scale $\mathbb{T}$.

The thesis is devoted to the following:

1. Illustrate the new theory of Stefan Hilger through a general introduction to the theory of dynamic equations on time scales,

2. Summarize some of the recent developments for the oscillation of second order impulsive differential equations and impulsive dynamic equations on time scales,

3. Establish some new sufficient conditions to ensure that all solutions of second order impulsive dynamic equations discussed in the thesis on unbounded time scales are oscillatory,

4. Give some examples to illustrate the importance of our results.

This thesis consists of six chapters:-

Chapter 1 contains the basic concepts of theory of functional differential equations and some preliminary results of the oscillation theory of second order impulsive differential equations.

In Chapter 2, An introduction to theory of dynamic equations on time scales, differentiation, integration and various properties of the exponential function on arbitrary time scale are added. Furthermore, the most important studies for the oscillation theory of second order impulsive dynamic equations on time scales are presented.

In Chapter 3, Some new oscillation criteria for the second-order nonlinear impulsive delay dynamic equation

$$\begin{cases} (r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t))^{\Delta}+p(t)f(x(\tau(t)))=0, \quad t\neq\theta_k, \ k=1,2,..., \\ \Delta(r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t))_{|t=\theta_k}+b_kh(x(\tau(\theta_k)))=0 \end{cases}$$

on a time scale $\mathbb{T}$ are established. Our results generalize and extend some pervious results [13, 36, 38, 40] and can be applied to some oscillation problems that not discussed before. These results extend the known results for the dynamic equations with and

without impulses.
These results of this chapter are published in:
Electronic Journal of Mathematical Analysis and Applications, accepted. [4].

In Chapter 4, The Riccati transformation technique is utilized to establish some new oscillation criteria for the second-order nonlinear impulsive dynamic equation with damping term

$$\begin{cases} (r(t)(x^{\Delta}(t))^{\alpha})^{\Delta} + q(t)(x^{\Delta}(t))^{\alpha} + f(t, x^{\sigma}(t)) = 0, \ \ t \neq t_k, \ k = 1, 2, ..., \\ x(t_k^+) = g_k(x(t_k^-)), \ \ x^{\Delta}(t_k^+) = h_k(x^{\Delta}(t_k^-)), \qquad k = 1, 2, ..., \\ x(t_0^+) = x_0, \ \ x^{\Delta}(t_0^+) = x_0^{\Delta}, \end{cases}$$

on a time scale $\mathbb{T}$, where α is the quotient of odd positive integers. Our results generalize and extend some pervious results [17, 18, 19, 20] and can be applied to some oscillation problems that not discussed before.
These results of this chapter are published in:
Acta Mathematica Universitatis Comenianae, 88 (2019), 1-16. [3].

In Chapter 5, Some new oscillation criteria for the second-order nonlinear impulsive dynamic equation of the form

$$\begin{cases} (r(t)g(x^{\Delta}(t)))^{\Delta} + f(t, x^{\sigma}(t)) = G(t, x^{\sigma}(t)), \ \ t \neq t_k, \ k = 1, 2, ..., \\ x(t_k^+) = \xi_k(x(t_k^-)), \ \ x^{\Delta}(t_k^+) = h_k(x^{\Delta}(t_k^-)), \ \ k = 1, 2, ..., \\ x(t_0^+) = x_0, \ \ x^{\Delta}(t_0^+) = x_0^{\Delta}, \end{cases}$$

on a time scale $\mathbb{T}$ are presented. Our results generalize and extend some pervious results [17, 18, 19, 27]. These results are published in:
Journal of Analysis and Number Theory, 5 (2017), 147-154. [2]

In Chapter 6, Some new oscillation criteria for the second-order nonlinear impulsive delay dynamic equation with perturbation

$$\begin{cases} (r(t)\psi(x(t))|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t))^{\Delta} + Q(t, x(\tau(t))) = P(t, x(\tau(t)), x^{\Delta}(t)), \ t \neq \theta_k, \ k = 1, 2, ..., \\ \Delta(r(t)\psi(x(t))|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t))_{|t=\theta_k} + Q_k(\theta_k, x(\tau(\theta_k))) = 0, \end{cases}$$

on a time scale $\mathbb{T}$ are established. Our results generalize and extend some pervious results [33, 37] and can be applied to some oscillation problems that not discussed before. These results extend the known results for the dynamic equations with and without impulses.

The results of this chapter are submitted for publication.

Chapter 1

Preliminaries

This chapter is considered as a background for the studies included in this thesis. We give a short survey of some definitions and theories used through this thesis.

1.1 Initial Value Problems

In this section, we give the definitions of ordinary and functional differential equations.

Definition 1.1.1

An ordinary differential equation (ODE) is a relation that contains function of only one independent variable, and one or more of its derivatives with respect to the variable.

Definition 1.1.2

A functional equation (FE) is an equation involves an unknown function for different argument values. The difference between the argument values of the unknown function and t in the FE are called argument deviations (see Saker [34]).

The equations $x(3t) + 4t^3x(6t) = 4$ and $x(t) = e^t x(t+1) - [x(t-3)]^2$ are examples of FEs.

Remark 1.1.1

If all argument deviations are constants (as in the second equation of the above example), then the FE is called a difference equation.

Combining definition 1.1.1 and definition 1.1.2, we obtain the definition of functional differential equation (FDE), or equivalently, differential equation with deviating argument as follows:

Definition 1.1.3

A functional differential equation is an equation contains an unknown function and some of its derivatives for different argument values. The order of a FDE is the order of the highest derivative of the unknown function. So, a FE may be regarded as a FDE of order zero (see Saker [34]).

Definition 1.1.4

The ordinary differential equation

$$x^{'}(t) = f(t, x) \tag{1.1}$$

together with the equation

$$x(t_0) = x_0 \tag{1.2}$$

is called an initial value problem (Eq. (1.2) is called an initial condition).

It is well known that under certain assumptions on f the initial value problem (1.1) and (1.2) has the unique solution,

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))ds \qquad for\ t \geq t_0 \tag{1.3}$$

Definition 1.1.5

The differential equation of the form

$$x^{'}(t) = f(t, x(t), x(t-\tau)) \qquad with \qquad \tau > 0\ \ and\ \ t \geq t_0, \tag{1.4}$$

in which the right-hand side depends on the instantaneous position $x(t)$ and the position at τ units back $x(t-\tau)$, is called an ordinary differential equation with delay or a delay differential equation.

Whenever necessary, we shall consider the integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\tau))ds, \tag{1.5}$$

equivalent to (1.4). Eq. (1.4) with the initial condition

$$x(t) = \varphi(t) \qquad for\ all\ \ t \in E_{t_0}, \tag{1.6}$$

where φ is a known initial function on the initial set E_{t_0}, t_0 is an initial point is known as the initial value problem for a delay differential equation. Usually, it is assumed that $\varphi(t_0 + 0) = \varphi(t_0)$. By a one-sided derivative, we mean the derivative at one of the endpoint of an interval. Under general assumptions, the existence and uniqueness of the solution of the initial value problem (1.4) and (1.6) is given in Ladas [22]. The solution is sometimes denoted by $x(t, \varphi)$. In case of variable delay $\tau = \tau(t) > 0$, the initial set E_{t_0} has the form:

$$E_{t_0} = t_0 \cup \{t - \tau(t) : t - \tau(t) < t_0, t \geq t_0\}.$$

If it is required to determine the solution on the interval $[t_0, T]$, then the initial set is

$$E_{t_0 T} = t_0 \cup \{t - \tau(t) : t - \tau(t) < t_0, t \geq t_0 \leq T\}.$$

Example 1.1.1

Consider the equation

$$x^{'}(t) = f(t, x(t), x(t - \cos^2 t)),$$

$t_0 = 0$, $E_0 = [-1, 0]$, *and the initial function φ must be given on the interval* $[-1, 0]$ *(see Agarwal et al. [1]).*

Definition 1.1.6

A point x of a set S is called an isolated point of S if there exists a neighborhood of x contains no points of S other than x itself.

Now, consider the differential equation of order n with l deviating arguments,

$$\begin{aligned} x^{(m_0)}(t) = f(t, x(t), ..., x^{(m_0-1)}(t), x(t-\tau_1(t)), ..., x^{(m_1-1)}(t-\tau_1(t)), ..., \\ x(t-\tau_l(t)), ..., x^{(m_l-1)}(t-\tau_l(t))), \end{aligned} \tag{1.7}$$

where the deviations $\tau_i(t) > 0$, and $n = \max\{\max_{1\le i\le l}(m_i - 1), m_0\}$. In order to formulate the initial value problem for (1.7), we assume that t_0 is the given initial point and for each deviation $\tau_i(t)$, the initial set $E_{t_0}^{(i)}$ is given by

$$E_{t_0}^{(i)} = t_0 \cup \{t - \tau_i(t) : t - \tau_i(t) < t_0, t \ge t_0\}.$$

We denote $E_{t_0} = \cup_{i=1}^{l} E_{t_0}^{(i)}$. On E_{t_0}, continuous functions φ_k, $k = 0, 1, ..., \lambda$, must be given with $\lambda = \max_{1\le i\le l}(m_i - 1)$. In applications, we often (not generally) consider the initial conditions,

$$\varphi_k(t) = \varphi_0^{(k)}(t) \qquad for\ k = 0, 1, ..., \lambda.$$

For the nth order differential equation, there should be given initial values $x_0^{(k)}$, $k = 0, 1, 2, ..., n-1$. Now let $x_0^{(k)} = \varphi_k(t_0)$, $k = 0, 1, 2, ..., \lambda$. If $\lambda < n - 1$, then the numbers $x_0^{(\lambda+1)}, ..., x_0^{(n-1)}$ must be given. Also if the point t_0 is an isolated point of E_{t_0}, then $x_0^{(0)}, ..., x_0^{(n)}$ must be given. For (1.7), the basic initial value problem consists of the determination of an $(n-1)$ times differentiable function x that satisfies (1.7) for $t > t_0$,

$$x^{(k)}(t_0 + 0) = x_0^{(k)}, \qquad k = 0, 1, 2, ..., n-1$$

and

$$x^{(k)}(t - \tau_i(t)) = \varphi_k(t - \tau_i(t)) \qquad for\ \ t - \tau_i(t) < t_0,$$

where $k = 0, 1, 2, ..., \lambda$ and $i = 0, 1, 2, ..., l$. At the point $t_0 + (k-1)\tau$ the derivative $x^{(k)}(t)$ is in general discontinuous, but the derivatives of lower order are continuous.

Example 1.1.2

For the equation

$$x^{''}(t) = f(t, x(t), x^{'}(t), x(t - \cos^2(t)), x(\frac{t}{2})). \tag{1.8}$$

In this case, we have $n = 2$*,* $l = 2$ *and* $\lambda = 0$*. For* $t_0 = 0$*, the initial sets are* $E_0^1 = [-1, 0]$*, and* $E_0^{(2)} = 0$*. Hence* $E_0 = [-1, 0]$*. On* E_0*, the initial function* φ_0 *is given. Also, the initial values* $x_0^{(0)} = \varphi_0(0)$*, and* $x_0^{(1)} = \varphi_1(0)$ *are given numbers (see Agarwal et al. [1]).*

For (1.7) a classification method was proposed by $Kamensk\check{i}\check{i}$ *[24]. Let* $\beta = m_0 - \lambda$*. If* $\beta > 0$*, (1.7) is called an equation with retarded arguments or with delay. If* $\beta = 0$*, (1.7) is called an equation of neutral type. If* $\beta < 0$*, it is called an equation of advanced type.*

Example 1.1.3

The equations

$$x^{'}(t) + a(t)x(t - \tau) = 0 \qquad \text{with} \qquad \tau > 0,$$

$$x^{'}(t) + a(t)x(t + \tau) = 0 \qquad \text{with} \qquad \tau > 0,$$

and

$$x^{'}(t) + a(t)x(t) + b(t)x^{'}(t - \tau) = 0 \qquad \text{with} \qquad \tau > 0,$$

are of retarded type $(\beta = 1)$*, advanced type* $(\beta = -1)$*, and neutral type* $(\beta = 0)$*, respectively (see Agarwal et al. [1]).*

The above classification is incomplete. For example, the equation

$$x^{'}(t) + ax(t - \tau) + ax(t + \sigma) = 0, \qquad \tau > 0,\ \sigma > 0 \tag{1.9}$$

does not belong to the above three classes.
Sometimes Eq. (1.9) is called an equation of mixed type.

1.2 Introduction to Delay-Differential Equations

Delay-differential equations (DDEs) form an important class of dynamical systems. They are often arise in either nature or technological control problems. In these systems, a controller monitors the state of the system, and makes an adjustment to the system based on its observations. Since these adjustments can never be made instantaneously, a delay arises between the observation and the control action.
In mathematics, delay differential equations (DDEs) are considered as differential equations in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times.
There are different kinds of delay-differential equations, such kinds are:
1- Continuous delay:

$$\frac{d}{dt}x(t) = f\big(t, x(t), \int_{-\infty}^{0} x(t+\tau)d\mu(\tau)\big).$$

2- Discrete delay:

$$\frac{d}{dt}x(t) = f(t, x(t), x(t-\tau_1), x(t-\tau_2), ..., x(t-\tau_m)),\ for\ \tau_1 > ... > \tau_m \geq 0.$$

3- Linear with discrete delays:

$$\frac{d}{dt}x(t) = A_0x(t) + A_1x(t-\tau_1) + ... + A_mx(t-\tau_m),$$

where $A_0, ..., A_m \in \mathbb{R}^{n\times n}$.
4- Pantograph equation:

$$\frac{d}{dt}x(t) = ax(t) + bx(\lambda t),$$

where a, b and λ are constants and $0 < \lambda < 1$.
In the following, we study equations with discrete delays.
In order to solve a delay equation, we need to consider the earlier values of x at each time step. Therefore, we need to specify an initial function which gives the behavior