Projection methods in different Banach-like spaces

Thesis By:

Moustafa Muhammed Zakaria Muhammed

(B.Sc. Degree in Mathematics – very good - June 2008)

(M. Sc. Degree in Pure Mathematics- October 2014)

(Teacher assistant at the Dept. of Math., Faculty of Science, Ain Shams University)

For the Degree of Doctor of Philosophy in Science
Pure Mathematics

Supervised By:

Prof. Dr. Nashat Faried Mohamed Fathi

Professor of Pure Mathematics
Mathematics Department
Faculty of Science
Ain Shams University

Dr. Hany Abd-Elnaim Mostafa Elsharkawy

Lecturer of Pure Mathematics
Mathematics Department
Faculty of Science
Ain Shams University

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Faculty of Science
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Summary

In a uniformly convex Banach space, the existence and uniqueness of a nearest point from a point outside a given subset are guaranteed when the given subset is nonempty, closed and convex, which is so called the projection theorem.

Faried and El-Sharkawy [12] extended this fact to the complete countably normed space which is a linear space equipped with a countable number of pair-wise compatible norms. They required that the completion (of the space equipped with each one of all its norms) to be uniformly convex and the convex subset to be closed with respect to all norms.

In this work, we prove the existence of a common nearest point (in all norms) from a point outside a nonempty subset of a countably normed space if this subset is compact with respect to all norms. We also prove the uniqueness of that common nearest point if the completion of the space equipped with only one of its norms is uniformly convex, see chapter 2.

Also we extend the projection theorem to the complete countably seminormed space (Fréchet space). In order to do this, we have to introduce a definition of uniformly convex seminormed space, projection theorem in seminormed space, a new vision of

the completion of countably seminormed space and a definition of uniformly convex countably seminormed space, see chapter 3.

This Ph. D. thesis is organized as follows:

1. Introduction, we show some importance of the projection theorem in Hilbert space and uniformly convex Banach space, and we give a general view of what we have been done in this thesis.

2. In chapter #1, we give a summary of semimetric and metric spaces [19], Banach space, Hilbert space [8, 22] and convex and uniformly convex Banach spaces [8, 9, 10, 1] almost of the needed details are available needed in this thesis.
   In section 1.7, we give a brief history of the development of the projection theorem in Hilbert space [8], reflexive and convex Banach space [1] and uniformly convex Banach space [8, 9, 1] which is very important in our work.

3. In chapter #2, we study countably normed space [14], completion of countably normed space [14], uniformly convex countably normed space [12], a projection theorem in countably normed space [12] and prove a new version of a projection theorem in countably normed space, **accepted** in [25].

4. In chapter #3, we give new definitions and prove new results **accepted** in [13]. In fact, we define new types of spaces, the so called *uniformly convex countably seminormed spaces*. We discuss some geometric properties for this new space and we prove the existence and uniqueness of the nearest point problem (the projection theorem) in a uniformly convex countably seminormed spaces.

Introduction

The projection theorem in Hilbert spaces (which is if K is a nonempty, closed and convex subset of a Hilbert space H, then for each $x \in H$ there exists a unique $\bar{x} \in K$ such that $\|x - \bar{x}\| = \inf_{y \in K} \|x - y\|$) is used in many applications, for example, in Optimization theory, fixed points theory and partial differential equations.

After J. A. Clarkson [10] defined the uniformly convex Banach space, they extended the projection theorem to the uniformly convex Banach space.

According to the projection theorem, they introduced the notion of *metric projection* which solved some fixed point problems for a non-self contraction mapping which maps a nonempty, closed and convex subset of a Banach space E into E [5, 2, 3, 4].

Faried and El-Sharkawy [12] defined a uniformly convex countably normed space, and they extended the projection theorem to a uniformly convex countably normed space.

In this thesis, we get another version of the projection theorem in the countably normed space which guarantees the existence and uniqueness of the common nearest point from a point outside a nonempty compact subset with respect to all norms requiring only that the completion of the space equipped with just one of its norms is uniformly convex. Also we give a more general theorem which proves the existence of a nearest point if the subset is nonempty and compact with respect to all norms.

We have generalized some nice geometric properties and definitions from countably normed spaces into countably seminormed spaces. Such as the definitions of *uniformly convex* and which play a very important role in the theory of normed spaces (see section 1.7). Also, we generalize the projection theorem to countably seminormed spaces (see chapter 3) which is very important in the theory of fixed point of noun self mapping.

Chapter 1

Some basics and metrizable spaces

In this chapter, we introduce the needed basics for the thesis such as semimetric, metric space, completion of metric space, seminomed, normed, inner product and Hilbert spaces. The details of these can be found in the books [19, 8, 9, 23, 22, 7, 17, 16, 21, 6, 24].

1.1 Semimetric and metric spaces

Definition 1.1.1 *(Semimetric and metric)[19, 23]*
A semimetric d on a nonempty set X is a mapping from $X \times X$ into the nonnegative half real line $\mathbb{R}^+$ with the following properties:
(1) $d(x, y) \geq 0$ *for all* $x, y \in X$.
(2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$.
(3) $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.

A semimetric is called a metric if $d(x, y) = 0$ *if and only if* $x = y$ *for all* $x, y \in X$.

If d is a semimetric (metric) on a nonempty set X, then the *pair (X, d) is called a semimetric (metric) space.*

The metric d is said to be translation invariant on a vector space X if the following condition is verified:
(4) $d(x, y) = d(x - z, y - z)$ for all $x, y, z \in X$
or that, for all pairs of points $x, y \in X$, $d(x, y) = d(x - y, 0)$.

Example 1.1.2 *[19]*
For any set X (with two or more distinct points), define $d : X \times X \to \mathbb{R}$ by $d(x, y) = 0$ for all $x, y \in X$. Then d is a semimetric but not a metric (since for $x \neq y \in X, d(x, y) = 0$). Clearly, d is a metric on X if and only if X is a singleton.

Example 1.1.3 *[19]*
The set of real numbers $\mathbb{R}$ with $d(x, y) = \frac{|x-y|}{1+|x-y|}$ form a metric space.

Example 1.1.4 *[19]*
The set s of all sequences of real numbers with

$$d(\{x_i\}, \{y_i\}) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|}$$

form a metric space.

Let (X, d) be a semimetric space. Define a relation on X such that $x\ R\ y$ if $d(x, y) = 0$, then this is equivalent relation. Hence, let $X/\ker d$ be the set of all equivalence classes $[x]$ such that $x \in X$ and assume $\bar{d}$ on $X/\ker d$ such that $\bar{d}([x], [y]) = d(x, y)$.

Proposition 1.1.5
$(X/\ker d, \bar{d})$ is a metric space.

Proof:
$\bar{d}$ is well defined: *Let $x_1, x_2, y_1, y_2 \in X$ such that $[x_1] = [x_2]$ and*

$[y_1] = [y_2]$.
We want to prove that $\bar{d}([x_1],[y_1]) = \bar{d}([x_2],[y_2])$.
Since $x_1 \; R \; x_2$ and $y_1 \; R \; y_2$, then $d(x_1,x_2) = 0 = d(y_1,y_2)$.
So, $\bar{d}([x_1],[y_1]) = d(x_1,y_1) \leq d(x_1,x_2) + d(x_2,y_2) + d(y_2,y_1) = d(x_2,y_2) = \bar{d}([x_2],[y_2])$. By the same way $\bar{d}([x_2],[y_2]) \leq \bar{d}([x_1],[y_1])$.
$\bar{d}$ is a metric: We check that if $\bar{d}([x],[y]) = 0$, then $[x] = [y]$, all other conditions are easy to verify. Now, let $\bar{d}([x],[y]) = 0$, then $d(x,y) = 0$. Hence $[x] = [y]$.

Definition 1.1.6 *(Metric space associated with semimetric space)*
For a semimetric space (X,d) there is a metric space $X/\ker d$ with the metric $\bar{d}([x],[y]) = d(x,y)$ called the associated metric space with the semimetric space (X,d).

Example 1.1.7
The metric space associated with semimetric space $(\mathbb{R}^2,d)$ such that $d((x_1,x_2),(y_1,y_2)) = |x_1 - y_1|$ is $\mathbb{R}$ with the metric $\bar{d}(x,y) = |x-y|$.

Since every metric is a semimetric, then we change the general definitions from metric space to semimetric space. This is required to prove our main results in seminormed and countably seminormed spaces (see chapter 3).

Definition 1.1.8 *(Ball and sphere)[19]*
Let (X,d) be a semimetric space. Given a point $x_0 \in X$ and a real number $r > 0$,

- $B(x_0,r) = \{x \in X : d(x,x_0) < r\}$ *(Open ball)*
- $\bar{B}(x_0,r) = \{x \in X : d(x,x_0) \leq r\}$ *(Closed ball)*
- $S(x_0,r) = \{x \in X : d(x,x_0) = r\}$ *(Sphere)*

In all three cases, x_0 is called the center and r the radius.

Definition 1.1.9 *(Interior and limit point)[19]*
A point x_0 is called an interior point for a subset A in a semimetric space (X, d) if there exists $B(x_0, r)$ with $r > 0$ such that $B(x_0, r) \subset A$. A point x_0 is called a limit point for a subset A in a semimetric space (X, d) if each open ball of x_0 contains a point y of the subset A other than x_0 itself. i.e.,

$$\forall B(x_0, r) : B(x_0, r) \backslash \{x_0\} \bigcap A \neq \emptyset.$$

Definition 1.1.10 *(Open and closed set)[19]*
A subset O of a semimetric space X is open if all its points are interior. A subset K of X is closed if it contains all its limit points.

Proposition 1.1.11 *[19]*
A subset O of a semimetric space X is open if and only if its complement O^c is closed.

Definition 1.1.12 *(Derived set and closure)[19]*
Let A be a subset of a semimetric space X. The derived set A' is the set of all limit points of A. The closure of the set A denoted by $\bar{A} = A \cup A'$.

Lemma 1.1.13 *[19]*
$\bar{A}$ of any subset A of a semimetric space is closed.

Definition 1.1.14 *(Dense subset)[19]*
A subset A of a semimetric space X is called every where dense if $\bar{A} = X$.

Definition 1.1.15 *(Bounded set and diameter of a set)[19]*
A set A in a semimetric space (X, d) is bounded if there exists $r > 0$ such that $d(x, y) \leq r$ for all $x, y \in A$.
Equivalent to say A is bounded if its diameter

$$\delta(A) = \sup_{x,y \in A} d(x, y)$$

is finite.

Definition 1.1.16 *(Convergent and Cauchy sequence)[19]*
Let (X, d) be a semimetric space.

1. *A sequence $\{x_i\} \subset (X, d)$ is said to be convergent to $x \in X$ if, given any $\epsilon > 0$, there exists an integer i_0 such that $d(x_i, x) < \epsilon$ for all $i \geq i_0$.*
   In this case, we write $\lim_{i \to \infty} x_i = x$ or simply $x_i \to x$.

2. *The sequence $\{x_i\}$ is said to be a Cauchy sequence if, given any $\epsilon > 0$, there exists an integer i_0 such that $d(x_i, x_j) < \epsilon$ for all $i, j \geq i_0$.*

The basic difference between semimetric and metric spaces is the uniqueness of a limit of a sequence, where semimetric space has no uniqueness of limit, but metric space has uniqueness of limit.

Lemma 1.1.17 *[19]*
Let (X, d) be a metric space.

1. *A convergent sequence in X is bounded and its limit is unique.*

2. *A Cauchy sequence in X is bounded.*