**Faculty of Education**
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# On the Simulation of Differential Transform Method using Pade′ Approximation for Studying Some Mathematical Models concerned with the Heat Transfer and Stability of Fluids

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# Summary

# Introduction and Basics

This chapter divided into two parts, first part presents the methods which used in the current thesis to solve the highly nonlinear problems of Non-Newtonian fluids flow. Second part offers a general introduction to the fluid mechanics and its important roles in many engineering, physiological and medical applications.

## Part 1 (Numerical analysis)

## 1.1 Introduction

Numerical analysis is the area of mathematics and computer science that creates analyzes and implements algorithms for solving numerically the problems of continuous mathematics. Such problems originate generally from real-world applications of algebra, geometry and calculus. These problems involve variables which vary consciously and occur throughout the natural science, social sciences, engineering, medicine and business. During the past half-century, the growth in power and availability of digital computers has led to an increasing use of realistic mathematical models in science and engineering. The formal academic area of numerical analysis varies from quite theoretical mathematics studies to computer science issues **[1]**. Recently, the growth in importance of using computers to carry out numerical procedures in solving mathematical models of the world an area known as scientific computing or computational science. This area looks at the use of numerical analysis from computer science perspective. It is concerned with using the most powerful tools of numerical analysis, computer graphics, symbolic mathematical computations and graphical user interfaces to make it easier for user to set up solve and intercept complicated mathematical models of real world **[2]**.

## 1.2 Numerical analysis with computer software

Numerical analysis and mathematical modelling have become essential in many areas to our life. Sophisticated numerical analysis software is being embedded in popular software packages, e.g. spreadsheet programs, allowing many people to perform modelling even when they are unaware of the mathematics involved in the process. This requires creating reliable, efficient and accurate numerical analysis software. It requires designing problem solving environments (PSE) in which it is relatively easy to model a given situation. The (PSE) for a given problem area is usually based on excellent theoretical mathematical models made available to the user through a convenient graphical user interface. Such software tools are well - advanced in some areas, e.g. computer aided design of structure, while other areas are still grappling with the more basic problems of creating mathematical models and accompanying tools for their solution, e.g. atmospheric and physiological models.

## 1.3 Ordinary differential equations

### 1.3.1 Boundary – value problems

In another class of fluid dynamic problems solution is to be found that satisfies not only the differential equation throughout some domain of its independent variable but also some conditions on boundaries of that domain. These problems are called boundary value problem (BVP). We consider here the numerical solution of a linear second order ordinary differential equation of the form:

$$\frac{d^2 f}{dx^2} + A(x)\frac{df}{dx} + B(x)f = D(x). \qquad (1.1)$$

For the domain $x_{min} \leq x \leq x_{max}$, subject to the boundary conditions that the values of $f$ are given at the end points of that range, at $x_{min}$ and $x_{max}$. The method developed here will be modified later for solving problems involving derivatives boundary conditions **[3]**.

## 1.4 Partial differential equation

A partial differential equation (PDE) is an equation for an unknown function of several independent variables (functions of these variables) that relates the value of the function and its derivatives of different orders. An ordinary differential equation (ODE) is a differential equation in which the functions that appear in the equation depend on a single independent variable. A partial differential equation is a function of multiple independent variables and their partial derivatives **[3]**. We classify the

## 1.5 Classification of partial differential equations

The most general form of a linear, second orders PDEs in a two independent variables $x, y$ and $u(x, y)$ is:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0, \qquad (1.2)$$

Where A, B, C, D, E, F and G are constants. The equation is called:

- Elliptic for $B^2 - AC < 0$, such as Laplace or Poisson's equation.

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial y^2} = f(x, y), \qquad (1.3)$$

- Parabolic for $B^2 - AC = 0$, such as Fourier heat equation

$$\frac{\partial^2 \varphi}{\partial x^2} - k\frac{\partial \varphi}{\partial t} = f(x,t), \tag{1.4}$$

- Hyperbolic for $B^2 - AC > 0$, such as the wave equation

$$\frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = f(x,t), \tag{1.5}$$

## 1.6 Parametric Numerical Deformation Solve (ND Solve)

The Mathematica function Parametric ND Solve is a general numerical differential equation solver. It can handle a wide range of ordinary differential equations (ODEs) as well as some partial differential equations (PDEs). In a system of ordinary differential equations there can be any number of unknown functions $x_i$, but all of these functions must depend on a single "independent variable" $t$, which is the same for each function. Partial differential equations involve two or more independent variables. Parametric ND Solve can also solve some differential-algebraic equations (DAEs), which are typically a mix of differential and algebraic equations [4, 5].

| | |
|---|---|
| $\boldsymbol{ParametricNDSolve}[\{\boldsymbol{eq1}, \boldsymbol{eq2}, \ldots\}, \boldsymbol{u}, \{\boldsymbol{t}, \boldsymbol{t_{min}}, \boldsymbol{t_{max}}\}]$ | Find a numerical solution for the function $\boldsymbol{u}$ with time $\boldsymbol{t}$ in the range $\boldsymbol{t_{min}}$ to $\boldsymbol{t_{max}}$. |
| $\boldsymbol{ParametricNDSolve}[\{\boldsymbol{eq1}, \boldsymbol{eq2}, \ldots\}, \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots\}, \{\boldsymbol{t}, \boldsymbol{t_{min}}, \boldsymbol{t_{max}}\}]$ | Find a numerical solution for several functions $u_i$. |

ParametricNDSolve represents solutions for the functions $x_i$ as interpolating function objects. The interpolating function objects provide approximations to the xi over the range of values $\boldsymbol{t_{min}}$ to $\boldsymbol{t}_{max}$ for the independent variable $t$. In general, Parametric ND Solve finds solutions iteratively. It starts at a particular value of $t$, and then takes a sequence of steps, trying eventually to cover the whole range $\boldsymbol{t_{min}}$ to $\boldsymbol{t}_{max}$. In order to get started, Parametric ND Solve has to be given appropriate initial or boundary conditions for the $x_i$ and their derivatives. These conditions specify values for$x_i(t)$, and perhaps derivatives $x_i{}'(t)$, at particular points $t$. When there is only one $t$ at which conditions are given, the equations and initial conditions are collectively referred to as an initial value problem. A boundary value problem occurs when there are multiple points $t$. ParametricNDSolve can solve nearly all initial value problems that can symbolically be put in normal form (i.e. are solvable for the highest derivative order), but only linear boundary value problems.

## 1.7 Differential transform method

The differential transform method (DTM) is a numerical as well as analytical method for solving integral equations, ordinary and partial differential equations. The method provides the solution in terms of convergent series with easily computable components. The concept of the differential transform was first proposed by Zhou **[6]** and its main application concern with both linear and nonlinear initial value problems in electrical circuit analysis. The DTM gives exact values of the $n^{th}$ derivative of an analytic function at a point in terms of known and unknown boundary conditions in a fast manner. This method constructs, for differential equations, an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor series method, which requires symbolic computations of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders. The DTM is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. Different applications of DTM can be found in **[7–12].** For convenience of the reader, we present a review of the DTM. The differential transform of the $k^{th}$ derivative of function $f(t)$ is defined as follows:

$$y(t, f, f', \ldots, f^{(n)}) = 0. \tag{1.6}$$

Subject to the initial equations

$$f^{(k)}(0) = d_k,\ k = 0, \ldots, n-1. \tag{1.7}$$

To demonstrate the differential transformation method (DTM) for solving differential equations, the basic definitions of differential transformation are introduced as follows. Let $f(t)$ be analytic in a domain $D$ and let $t = t_0$ represent any point in $D$. The function $f(t)$ is then represented by one power series whose centre is located at $t_0$. The differential transformation of the $k^{th}$ derivative of a function $f(t)$ is defined as the following:

$$F(k) = \left(\frac{1}{k!}\right)\left[\left(\frac{d^{(k)}f(t)}{dt^{(k)}}\right)\right]_{(t=t^0)},\ \forall t \in D. \tag{1.8}$$

And the inverse transformation of $F(k)$ can take the form

$$f(t) = \sum_{k=0}^{\infty} F(k)(t - t_0)^{(k)},\ \forall t \in D. \tag{1.9}$$

In fact, from Eq. (1.8) and (1.9), we obtain

$$f(t) = \sum_{k=0}^{\infty} \frac{(t-t_0)^{(k)}}{k!}\left(\frac{d^{(k)}y(t)}{dt^{(k)}}\right)_{t=t_0}, \forall t \in D. \tag{1.10}$$

Eq. (1.10) implies that the concept of differential transformation is derived from the Taylor series expansion. Form the definitions of (1.8) and (1.9), it is easy to prove that the functions comply with the following basic mathematics operations (see Table 1). In real applications, the function $f(t)$ is expressed by a finite series and (1.10) can be written as:

$$f(t) = \sum_{k=0}^{N} F(k)(t - t_0)^{(k)}, \ \forall t \in D. \tag{1.11}$$

Eq. (1.11) implies that $\sum_{k=N+1}^{\infty} F(k)(t - t_0)^{(k)}$ is negligibly small. The following table show that the transformation for some functions and relation by using differential transformation method.

**Table 1.1**
Operations of the one dimensional differential transform method which uses in this thesis.

| *Original function* | *Transformed function* |
|---|---|
| $f(t) = g(t) \mp h(t)$ | $F(k) = G(k) \mp H(k)$ |
| $f(t) = \alpha g(t)$ | $F(k) = \alpha G(k)$ |
| $f(t) = g(t)h(t)$ | $F(k) = \sum_{l=0}^{k} G(l)H(k-l)$ |
| $f(t) = \frac{dg(t)}{dt}$ | $F(k) = (k+l)G(k+l)$ |
| $f(t) = \frac{d^n g(t)}{dt^n}$ | $F(k) = \frac{(k+l)!}{k!} G(k+n)$ |
| $f(t) = u(t)v(t)w(t)$ | $F(k) = \sum_{l=0}^{k} \sum_{r=0}^{k-l} U(l)V(l)w(k-r-l)$ |
| $f(t) = t^n$ | $F(k) = \delta(k-n) = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$ |
| $F(t) = \left(\frac{dg(t)}{dt}\right)^2$ | $F(k) = \sum_{r=0}^{k} \begin{matrix} (r+1)(k-r+1) \\ G(r+1)G(k-r+1) \end{matrix}$ |
| $F(t) = \left(\frac{dg(t)}{dt}\right)^4$ | $F(k) = \sum_{r_3=0}^{k} \sum_{r_2=0}^{r_3} \sum_{r_1=0}^{r_2} \begin{matrix} (r_1+1)(r_2-r_1+1)(r_3-r_2+1)(k-r_3+1) \\ G(r_1+1)G(r_2-r_1+1)G(r_3-r_2+1)G(k-r_3+1) \end{matrix}$ |
| $F(t) = \frac{dg(t)}{dt}\frac{dh(t)}{dt}$ | $F(k) = \sum_{r=0}^{k} \begin{matrix} (r+1)(k-r+1) \\ G(r+1)H(k-r+1) \end{matrix}$ |
| $F(t) = \frac{dg^2(t)}{dt^2}\left(\frac{dg(t)}{dt}\right)^2$ | $F(k) = \sum_{r_2=0}^{k} \sum_{r_1=0}^{r_2} \begin{matrix} (r_1+1)(r_2-r_1+1)(k-r_2+1)(k-r_2+2) \\ G(r_1+1)G(r_2-r_1+1)G(k-r_2+2) \end{matrix}$ |
| $F(t) = \left(\frac{dg^2(t)}{dt^2}\right)^2$ | $F(k) = \sum_{r=0}^{k} \begin{matrix} (k-r+1)(k-r+2) \\ G(r)U(k-r+2) \end{matrix}$ |

## 1.8 Multi-step differential transform method

The DTM introduces a promising approach for many applications in various domains of science. However, DTM has some drawbacks. By using the DTM, we obtain a series solution, actually a truncated series solution. This series solution does not exhibit the real behaviors of the problem but gives a good approximation to the true solution in a very small region. For this purpose, researchers **[13-20]** have presented a new reliable algorithm of the DTM. The new algorithm, multi-step DTM, presented in this paper, accelerates the convergence of the series solution over a large region and improves the accuracy of the DTM. The validity of the modified technique is verified through illustrative examples of Lotka–Volterra, Chen and Lorenz systems.

Suppose $[0, T]$ is the interval over which we want to find the solution for a system of equations$(1.8 - 1.10)$. In actual applications of the DTM, the approximate solution for a system of equations can be expressed by the finite series

$$f(t) = \sum_{k=0}^{N} a_{(k)} t^{(k)}, \ t \in [0, T]. \tag{1.12}$$

The multi-steps approach introduces a new idea for constructing the approximate solution. Assume that the interval $[0, T]$ is divided into $M$ sub intervals $[t_{m-1}, t_m]$, $m = 1,2, \ldots, M$ of equal step size $h = \frac{T}{M}$ by using the nodes $t_m = mh$. The main ideas of the Multi-step DTM are as follows. First, we apply the DTM to a system of equations $(1.8 - 1.10)$ over the interval $[0, T]$ we will obtain the following approximate solution,

$$f_1(t) = \sum_{k=0}^{N} a_{1n} t^k, t \in [0, t_1], \tag{1.13}$$

Using the initial conditions $f^{(k)}(0) = C_k$ Form $m \geq 2$ and at each sub interval $[t_{m-1}, t_m]$ we will use the initial conditions $f_m^{(k)}(t_{m=1}) = f_{m=1}^{(k)}(t_{m=1})$ and apply the DTM to Eqs. $(1.8 - 1.11)$ over the interval$[t_{m=1}, t_m]$, where $t_0$ in Eq. (1.17) is replaced by $t_{m-1}$ the process is repeated and generates a sequence of approximate solution sum. $f_m(t), m = 1,2, \ldots, M$ for the solution $f(t)$.

$$f_m(t) = \sum_{k=0}^{N} a_{mk} \left(t - t_{\{m-1\}}\right)^2, t \in [t_m, t_{m-1}] \tag{1.14}$$

where $N = K.M$. In fact, the multi-step DTM assumes the following solution

$$u(t) = \begin{cases} u_1(t), & t \in [0, t_1] \\ u_2(t), & t \in [t_1, t_2] \\ \vdots & \\ u_M(t) & t \in [t_{M-1}, t_M] \end{cases} \tag{1.15}$$

The new algorithm, multi-step DTM, is simple for computational performance for all values of $h$. It is easily observed that if the step size $h = T$, then the multi-step DTM reduces to the classical DTM. As we will see in the next section, the main advantage of the new algorithm is that the obtained series solution converges for wide time regions and can approximate non-chaotic or chaotic solutions.

### 1.9 Finite difference method

The key to various numerical methods is to convert the partial differential equations that govern a physical phenomenon into a system of algebraic equations. Thus, it can be very efficiently solved numerically by many methods. For instances, direct elimination and iterative methods. Different techniques are available for this conversation. For the previous reasons, finite difference method has become more widely used in recent years. It is one of the most commonly used because the resulting solutions are accurate to any order **[21]**.

#### 1.9.1 Finite difference formulation

There are several approaches for the numerical solution of the fluid dynamic equations and most are based on a basic element. For example, the conservation equations can be applied to a fluid element by use of an integral or differential representation. Some methods that apply the integral form of the equation to a basic element are called finite volume. Another approach uses the finite difference approximation for the fluid dynamic equations. As noted, for simplicity, let us follow here this latter method **[22]**. The first step of the processes is to develop the numerical representation of the various terms (derivatives) in the fluid dynamic equations. Let us do this by applying Fig(1.1), where $u(x)$ is an analytic on the function along the coordinate $x$. Next, we select several nodal points on the function $u(x)$ separated by the distance $\Delta x$. The objective is then to relate the function to its derivatives, based on the function values at the nodal points. For small $\Delta x$, $u(x)$ can be expanded in a Taylor series about $x$(**[23]**):

$$u(x+\Delta x) = u(x) + \Delta x \frac{du}{dx} + \frac{(\Delta x)^2}{2!}\frac{d^2u}{dx^2} + \frac{(\Delta x)^3}{3!}\frac{d^3u}{dx^3} + \cdots. \qquad (1.16)$$

Fig(1.1) illustrating the method the method for approximation the derivatives of the function $u(x)$.

For example, we can find the first derivatives, $\frac{du}{dx}$, expressed in terms of the values at the nearby nodal points, by simple algebraic operation. Using first order terms from equation (1.16) and neglecting the higher order terms and solving for $\frac{du}{dx}$ we get:

$$\frac{du}{dx} = \frac{u(x+\Delta x) - u(x)}{\Delta x} + O(\Delta x), \qquad (1.17)$$