SOME MODELS OF COMBINATORIAL GAMES

A Thesis

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(Pure Mathematics)

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SUMMARY

Combinatorial games CGT are two-player game with perfect information and no chance. For example, the child's play TIC-TAC-TOE. In combinatorial game theory, we make analysis of the combinatorial game, we describe the game and we try to predict the winner for any game. Studying the winning strategy for the combinatorial games is the most important part of combinatorial game theory. In this thesis, we construct some models of combinatorial games.

CGT arise in relation to the theory of impartial games, in which any play available to one player must be available to the other as well. One such game is Nim, which can be solved completely. Nim is an impartial game for two players, and subject to the *normal play condition*, which means that a player who cannot move loses. In the 1930s, the Sprague–Grundy theorem [1] showed that all impartial games are equivalent to heaps in nim, this implies that major unifications are possible in games considered at a combinatorial level, in which detailed strategies matter, not just pay-offs.

In the 1960s, Elwyn R. Berlekamp, John H. Conway [2] jointly introduced the theory of a partisan game, in which the requirement that a play available to one player is available to both is relaxed. Their results were published in [2]. However, the first work published on the subject was Conway's book [3] , in which as been introduced the concept of surreal numbers and the generalization to games.

Combinatorial games are generally, by convention, put into a form where one player wins when the other has no moves remaining. It is easy to convert any

finite game with only two possible results into an equivalent one where this convention applies. One of the most important concepts in the theory of combinatorial games is that of the sum of two games, which is a game where each player may choose to move either in one of the two games any point in the game, and a player wins when his opponent has no move in either game. This way of combining games leads to a rich and powerful mathematical structure.

the thesis consists of four chapters which are organized as follows:

Chapter 1: Contains the basic definitions and the types of combinatorial games, impartial and partisan game, and normal and misere games. We we also give a brief description of the game of Nim, Nimble, Dawson's Kayles, silver dollar, two dimensional Nim, Wythoff's Game and Green Hachenbush. We also introduced some models of partisan games like Blue-Red Hackenbush, Blue-Red-Green Hackenbush, Col, Snort, and Domineering.

Chapter 2: In this chapter, we analyze the concepts of impartial games and partisan games and we investigate the difference between the analysis of the impartial games and the partisan games.

Chapter 3: This chapter is concerned with the game of Nim in the case of n-player and the alliance system of the players. Multi-player small Nim, the multi-player last Nim. We defined an alliance matrix and we called it the shifted standard alliance matrix and we studied the multi-player last Nim if the shifted alliance matrix is adopted. The main results of this chapter have been published in [4]

Chapter 4: In this chapter, we introduced an algorithm that helps in solving any multi-player last Nim for any alliance system, this algorithm helps us to know the winner of the game for any alliance system. The main results of this chapter has been published in [5]

Chapter 1

Basics Concepts of Combinatorial Games

1.1 Introduction

Combinatorial game theory is a part of mathematics science committed to concentrate the ideal procedure in perfect information data games where commonly two players are included. In a 2-man flawless data game, two players substitute moves until one of them cannot move at this turn. Among the games of this sort, as a non-comprehensive rundown, are Nim (Bouton [6], Fraenkel and Lorberbom [7], Flammenkamp [8], Holshouser [9], Albert and Nowakowski [10], Liu and Zhao [11]), End-Nim (Albert and Nowakowski [12]), and so on. Last Nim with two players presented by Friedman [13] is played with heaps of counters which are straightly requested. The two players alternate expelling any position whole number of counters from the last heap. Under normal play tradition, all P-positions of Last Nim with two players are those containing an odd number of heaps containing one counter.

1.1.1 Multi-player Combinatorial Game

Amid the most recent couple of years. The theory of 2-player perfect information games has been generally examined. Normally, it is important to sum up however much as could reasonably be expected of the theory of n-player games. In 2-player perfect information games, one can discuss what the result of the diversion ought to be, at the point when every player play it right, i.e. at the point when every player embraces an ideal methodology yet when there are multiple players', it may not bode well discuss the similar thing. For example, it might so happen that one of the players can help any of the players to win, yet at any rate, he himself needs to lose. Along these lines, the result of the game relies upon how alliances are shaped among the players, in past studies a few conceivable outcomes were researched: multi-player without alliance, multi-player with two alliances and multi-player with alliance system.

1.1.1.1 Multi-player Without Alliance

N-player Nim game has been submitted without alliance by Li [14]. Straffin [15] tried to classify the three players Nim game with somewhat restrictive assumption regarding the behavior of each player. This work investigated by Loeb [16] by introducing the concept of stable alliance (where an alliance member wins) the work is done by Propp [17] analyzed the conditions required to allow one player has a winning strategy against the combined force of the other. Cincotti [18] gave an analysis of n-player partisan games.

1.1.1.2 Multi-player With Two Alliances

In [19] [20] Kelly introduced one bonded Nim, denoted by OBN, which is considered as one pile Nim with two alliance and n-player, any player in this will help his alliance to win, under normal play the alliance will remove the last counter will win but under misere play, the alliance will remove the last counter will lose. The general structures of the two alliances were introduced by Zhao and Liu [21],

1.1.1.3 Multi-player With Alliance System

Krawec [22] considered that every player has a fixed set of an alliance, i.e. he will arrange according to preference and this alliance system will be known before beginning the game and this alliance system presented by a matrix. Krawec [23] improved a method of analyzing the impartial combinatorial game with n player and considered that one of the players play randomly with no strategy.

1.1.1.4 Multiple-player With Alliance System With Pass

Liu and Yang(2017) [24] studied the multiple-player Last Nim when the standard alliance matrix adopted in, this matrix we will explain it below, each of n players either removes any positive integer of counters from the last pile or makes a choice 'pass'. Once a 'pass' option is used, the total number s of passes decreases by 1. When all s passes are used, no player may ever 'pass' again. A pass option can be used at any time, up to the penultimate move, but cannot be used at the end of the game, he determined the game value of the game with a different number of piles and players.

1.2 What Is a Combinatorial Game

In any combinatorial game we have the following:

(i) Two players: Every combinatorial game is played by two players. And the two players usually distinguished Right and Left, or equally Blue and Red. Players may be called First and Second while analyzing impartial games, and there is no importance in distinguishing their identities, and the only difference is the order in which they make there moves. We will see a combination of these two options on the roles of the two players.

(ii) No chance: The factors of chance are neglected, the games are played with certain results for every move without the effect of coincidence. This condition not allowed in the games played with shuffled cards roulette, dice or any coincidence element of any kind, and that helps us to get the accuracy for the outcomes that can be deduced from the analysis of the game.

(iii) Perfect information: Two players have complete knowledge about the game and there is no secret moves and there is no hidden information in the game, the two players have all information about the game.

(iv) Alternating game: The two players are playing the game alternately, i.e. the first player makes his move then the second player makes his move and continues in this manner. Choosing the first player can be considered as the only allowed element of chance in combinatorial games.

(v) Absolute winner: There is only one winner for the combinatorial game, this means there is no draw. The options of the draw are ruled out by this condition.

(vi) Winning condition: The game is defined as reaching the terminal position in which the game ends by winning one of the two player this means that the combinatorial games are not infinite games but for every combinatorial game there is a terminal position.

In combinatorial games there are two types of games which called **partisan game** and **impartial game**, we will take about it as follows:

Definition 1.2.1. Iimpartial games, the two player have the same options, i.e. the two player have the same set of moves for both players. There is an only way to distinguish players is the order of their moves.

Definition 1.2.2. A partisan game is a game with separately specified moves for both players. These moves are distinguished by the direction of moves or different colors of counters. The analysis of this game then concerns the possibilities of both players by given positions and their advantage.

There are two kinds of winning conditions in impartial games and partisan games, in these kinds we can identify the winner player.

Definition 1.2.3. Normal play, the last player in this type of games who can make a move is the winner of the game.

Definition 1.2.4. Misère play, the last player in this type of games who can make a move is the loser of the game.

1.3 Examples of Impartial Games

1.3.1 One Pile Game

This game is the simplest example of an impartial game. The game is played by two players with a number of counters ordered in one row. The move of each player consists of reducing this row by one, two or three counters. The common example for the one pile game is the 21 game. The first player who makes the last move is considered the winner. The two players make their moves alternately.

The solution of the 21 game is the player who can make in the pile 4,8,12,16,20 counters are the players who has a winning strategy.

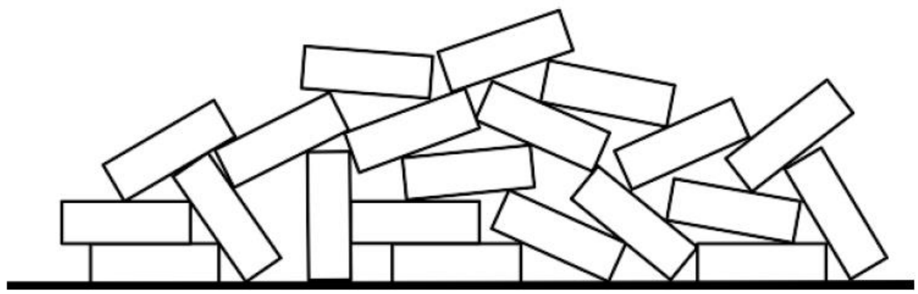

Figure 1.1: One pile game

1.3.2 The Game of Nim

This game is the best one to explain the theory concerning solving impartial games. The game is, similarly to the One pile game, played with the counters in rows with a difference of any number of piles of counters instead of only one. There is also a change in the definition of a legal move, which in this case consists of taking any number (greater than zero) of counters from one pile and one pile only. Same as in the One pile game losing player is the one, who is the first unable to take any counters, because there are none left by the opponent, and the other player is, naturally, a winner. After a long time of playing this game, a bright mind can easily find basic patterns of moves and states from which it is easy to guess the winner before the game ends, and so there are many commonly known disguises of this game with the purpose of confusing and entertaining both players at the same time.

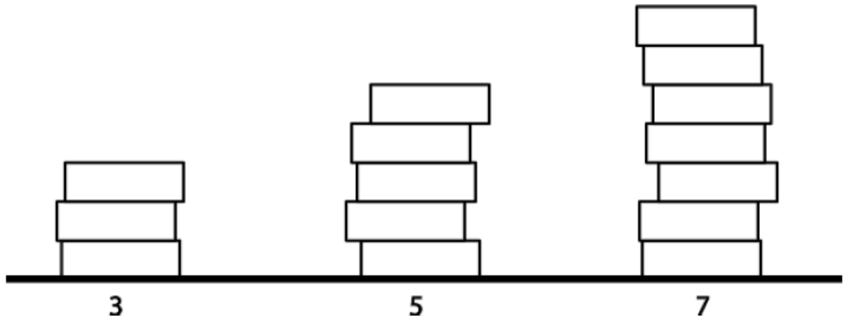

Figure 1.2: Game of Nim

1.3.3 Nimble

An alternation of Nim played with coins on a semi-infinite board of squares (for our purposes the border of this board will always be on the left side, which leaves the right one endless). The starting position of this game consists of a finite amount of coins randomly spread on the squares of this board. In each turn of the game, the player moves one of the coins leftwards. The goal of the game is to reach a position from which no other move is possible. In other words, the winning position of this game is having all coins positioned on the first square, so the other player has no move to play and loses the game.

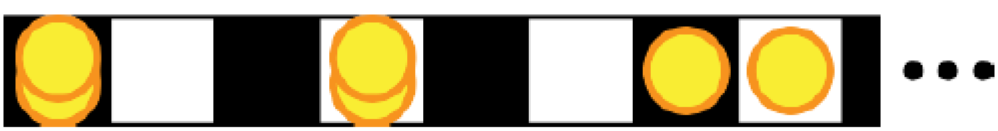

Figure 1.3: Nimble

1.3.4 Dowson's Kayles

Dawson's Kayles is played with several rows of boxes, such as those shown in Figure 1.4(a). On his turn, a player must remove exactly two adjacent boxes from one of the strips. For example, he might play as shown in Figure 1.4(b), splitting the strip of size 10 into two smaller strips and leaving the position (c). Eventually, all remaining boxes will be contained in strips of length one (or perhaps, no boxes will be left at all), after which no further moves are available.

There are two ways to play Dawsons's Kayles:

- In normal play, the player who makes the last move wins. This is the same convention used in NIM.
- In misere play, the player who makes the last move loses.

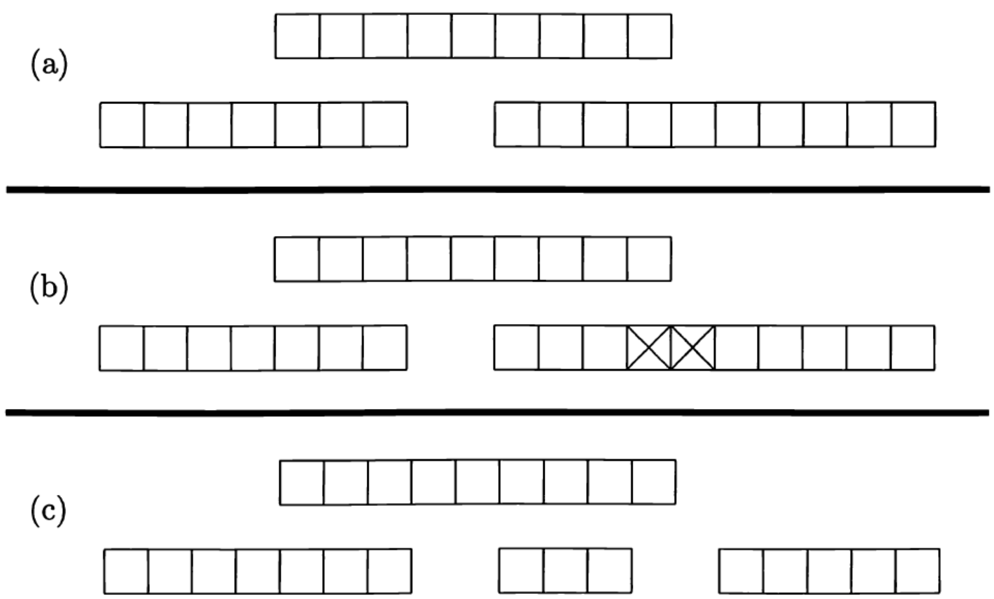

Figure 1.4: Dawson Kayles

1.3.5 The Silver Dollar Game

The silver dollar game is a game similar to the Nimble game. There are two types of the silver dollar games, the first is the silver dollar game without a silver dollar, the second is the silver dollar game with a silver dollar, the main difference is, as the name of the two types of the game implies.

The silver dollar game without silver dollar is a game similar to the Nimble game and has two added restrictions – no coin can move over any other coins, and no two or more coins can be placed in the same square of the board. It end if all coins are lined-up at the beginning of the board. With no empty space between the coins.

The silver dollar game with silver dollar only have the same restriction of a maximum of one coin in any square and this rule is not valid for the first

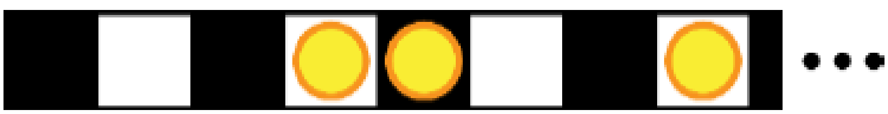

Figure 1.5: Silver dollar game without the silver dollar

square, this means that any number of coins can placed in the first square. Also no coins can move over any other coin. And there is a separated coined labeled as a silver dollar. This coin is the last coin on the board on the most right square of the board. The goal of this game is to take the silver dollar from the first square and takes it off the board. And this can be done if and only if all coins are placed there. And the game end if no player can make a move and the winner is the player who makes the last move.