Faculty of Education
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Spectral Properties and Entropy of a Two-Level Atom in a Squeezed Vacuum with an Exponential Decay of the Driving Field

A Thesis

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(Quantum Mechanics)

Submitted to:

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بِسْمِ اللهِ الرَّحْمنِ الرَّحِيمِ

قَالُواْ سُبْحَانَكَ لاَ عِلْمَ لَنَا إِلاَّ مَا عَلَّمْتَنَا إِنَّكَ أَنتَ الْعَلِيمُ الْحَكِيمُ

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Abstract

In this thesis the spectral properties, Boltzmann-Gibbs (BG) atomic entropy and the zero-absorption isolines of a qubit in an off-resonance squeezed vacuum (SV) reservoir with an exponential decay of a coherent driving field are studied. The model Bloch equations have time-dependent coefficients and hence have no exact solutions. Two different perturbative approaches are used to solve these equations in the two limiting cases, namely, the weak and strong field cases.

Having solved the Bloch equations, the instantaneous average atomic population, polarization and the BG atomic entropy are calculated in both limits of weak and strong field cases. Furthermore, the quantum regression theorem is used to obtain analytical expressions for the transient linear susceptibility in the normal vacuum (NV), thermal field (TF) and SV reservoir cases. Also, the zero-absorption isolines are identified.

Contents

Chapter 1

Introduction

1.1 Thesis outline

The thesis consists of four chapters. Their contents are as follows:

- Chapter 1

  This chapter represents an introduction for the thesis. The basic information and concepts needed in the thesis are introduced. First, briefs on quantum mechanics and quantum optics are presented. Second, the derivation of the optical Bloch equations for spin-1/2 system is reviewed. Third, states of the quantized radiation field are presented. Finally, the atomic entropy and the quantum regression theorem are summarized.

- Chapter 2

  In this chapter, the usual perturbation approach is adopted to solve the model Bloch equations for a weakly excited 2-level atom in an off-resonant SV with an exponentially decaying driving field up to the second order in the Rabi frequency of the driving field. Both the instantaneous mean atomic variables (population and polarization) and the BG atomic entropy are calculated. Effect of the SV parameters and the damping rate of the driving field are examined computationally. Entropy investigation shows that, the system is unstable for certain values of the SV phase parameter.

- Chapter 3

  An iterative approach is adopted to solve the model Bloch equations for a 2-level atom in a resonant SV with an exponentially decaying strong driving field. Both the instantaneous mean atomic variables and the BG atomic entropy are calculated. Effect of the SV parameters, the damping rate of the driving field, and the Rabi frequency are examined graphically. Atomic entropy is used as a measure of the system stability.

- Chapter 4

  Analytical expressions for the absorption-dispersion spectra for a weakly driven 2-level atom in a broadband SV with an exponentially decaying of the driving field are deduced. The atomic transition frequency and the central frequency of both the SV and the driving field are assumed to be different. For the resonant case, the absorption behaves as an even function in the NV, TF, and SV cases, while the dispersion behaves as an odd function in the NV, TF, and SV cases. The zero-absorption isolines in the (Δ, Ω)- plane are symmetric for resonant probe field but asymmetric for the Rabi-side probe field frequency (where Δ is the atomic detuning parameter and Ω is the Rabi frequency). The largest set of points, where absorption is zero, in the (Δ, Ω)-plane depends on the phase φ and on the SV detuning parameter.

1.2 Briefs on quantum mechanics

- Classical physics is controlled by two essential concepts. The first, is the concept of a particle which moves according to Newton's laws of motion. The second concept, is that of an electromagnetic wave that change according to Maxwell's laws of electromagnetism [1].

- The failure of classical physics to demonstrate several microscopic phenomena (such as, blackbody radiation, photoelectric effect, atomic stability, and atomic spectroscopy) had paved the way for seeking new ideas outside its purview [2–5]. The new theory, known as quantum mechanics. Briefly, quantum mechanics is the theory that depicts the dynamics of matter at the microscopic scale, $O(10^{-8})$ cm or less. It is the theoretical basis of modern physics that expounds the nature and behavior of matter and energy on the atomic and subatomic levels.

- Historically, there were two independent formulations of quantum mechanics, namely, matrix mechanics and wave mechanics [6–8]. The matrix mechanics (Heisenberg picture), was developed by Heisenberg in 1925. The starting point of Heisenberg's formulation is to find a theoretical foundation to Bohr's ideas. Matrix mechanics is very efficient in accounting for discrete quanta of light emitted or absorbed by the atom.

- The wave mechanics introduced by Schrödinger in 1926 is a generalization of the de Broglie postulate (wave-particle duality for both radiation and matter). This method depicts the dynamics of microscopic matter via a wave equation, known as the Schrödinger differential wave equation, instead of the matrix eigenvalue problem of Heisenberg. The solution of this equation gives the energy spectrum and the wave function of the system under consideration.

- These two different formulations, Schrödinger's wave formulation and Heisenberg's matrix approach, were proven to be equivalent [9]. The theory of quantum mechanics that Schrödinger and Heisenberg assumed works only for non-relativistic phenomenon, so this theory is called non-relativistic quantum mechanics. Combining the theory of special relativity with quantum mechanics, Dirac succeeded in extending quantum mechanics to the regality of relativity theory. The new theory, known as relativistic quantum mechanics, predicted the existence of a new particle, the positron [5].

1.3 Briefs on quantum optics

Optics is the branch of physics concerned with the nature of light, its propagation in different media and its interaction with different materials. Quantum is a discrete amount of energy absorbed or emitted by the atom in the form of a photon. Photons are introduced by Planck in his efforts to explain the phenomenon of blackbody radiation. Quantum optics is a branch of modern physics that deals with the application of quantum mechanics to phenomena involving light and its interaction with matter. Quantum optics deals with optical phenomena that can be explained by treating light as a stream of photons rather than an electromagnetic wave [10, 11].

In the gradual development of the theory of light, three general approaches are identified, namely, classical, semi-classical and quantum theories. Classical theories treat both atom and light classically. Semi-classical theories treat light as a classical electromagnetic wave and apply quantum mechanics to the atom. Quantum theories apply quantum concept to both atom and light.

The quantum optical approach is consistent with itself and with the full body of experimental data. Nonetheless, the semi-classical theories are suitable to deal with some phenomena like absorption of light by atoms, where the quantum mechanics is applied to the atoms and the light is treated as classical electromagnetic waves. There are some phenomena that classical theories cannot explain them (e. g. spontaneous emission and lamb shift). These phenomena required a quantum model of light [12].

1.4 States of the quantized radiation field

1.4.1 Number state

The quantized free electromagnetic field is an infinite collection of uncoupled harmonic oscillators, each of which is described by the Hamiltonian $\hat{H}_\lambda = \hbar\omega_\lambda(\hat{N}_\lambda + \frac{1}{2})$ [6, 15]. Since all the oscillators are uncoupled, only single oscillator will be studied and the mode index λ will be omitted.

To solve the eigenvalue problem for $\hat{H} = \hbar\omega(\hat{N} + \frac{1}{2})$, the eigenvalue problem for the number operator $\hat{N}|n\rangle = n|n\rangle$ must be solved firstly, where n is the eigenvalue of $\hat{N}$ and $|n\rangle$ is the corresponding eigenvector. The number operator $\hat{N} = \hat{a}^\dagger\hat{a}$ is a hermitian operator, therefore its eigenvalues n are real and its eigenvectors $\{|n\rangle\}$ form a complete orthogonal set of basis vectors in the n-representation. By using the argument that the norm of any Hilbert-space vector must be non negative, then the eigenvalues of the operator $\hat{N}$ are non-negative integers (i.e. $n = 0, 1, 2, 3, ...$).

The eigenvalues n may be regarded as the number of quanta of the harmonic oscillator associated with the radiation field under consideration. These quanta of energy $\hbar\omega$ are called photons. Accordingly, the number operator $\hat{N}$ and its eigenstate $|n\rangle$ are called the photon number operator and the photon number states or Fock States.

If the harmonic oscillator starts in the state $|n\rangle$ with n quanta and we operate with the non-hermitian operator $\hat{a}$ it generates the state $|n-1\rangle$ with $n-1$ quanta (where $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$) so the operator $\hat{a}$ is called annihilation or lowering operator; while if we operate with $\hat{a}^\dagger$ it generates the state $|n+1\rangle$ with $n+1$ quanta (where $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$), so $\hat{a}^\dagger$ is called creation or raising operator. The quadrature operators $\hat{X}_{1,2}$ are defined as follows [16]:

$$\hat{X}_1 = \frac{1}{2}(\hat{a}^\dagger + \hat{a}), \tag{1.1a}$$

$$\hat{X}_2 = \frac{i}{2}(\hat{a}^\dagger - \hat{a}). \tag{1.1b}$$

They are, essentially, the dimensionless versions of the position operator $\hat{q}$ and momentum operator $\hat{p}$, respectively, they obey the commutation relation $[\hat{X}_1, \hat{X}_2] = \frac{i}{2}$. For arbitrary chosen state, the fluctuation of the two quadrature operators satisfy the uncertainty principle

$$(\Delta\hat{X}_1)(\Delta\hat{X}_2) \geq \frac{1}{4}. \tag{1.2}$$

The expectation values of $\hat{X}_{1,2}$ and $\hat{X}_{1,2}^2$ in the number state $|n\rangle$ are

$$\langle\hat{X}_i\rangle = 0, \tag{1.3a}$$

$$\langle\hat{X}_i^2\rangle = \frac{1}{4}(2n+1), \qquad i = 1, 2, \tag{1.3b}$$

and the variances are

$$(\Delta\hat{X}_i)^2 = \langle\hat{X}_i^2\rangle - \langle\hat{X}_i\rangle^2 = \frac{1}{4}(2n+1), \qquad i = 1, 2. \tag{1.4}$$

Then

$$(\Delta\hat{X}_1)(\Delta\hat{X}_2) = \frac{1}{4}(2n+1) \geq \frac{1}{4}. \tag{1.5}$$

For $n > 0$ the number state is not a minimum uncertainty state but for $n = 0$ (vacuum state) is a minimum uncertainty state.

1.4.2 Coherent state

Coherent state is very useful in dealing with radiation problems. Coherent state is a minimum uncertainty state with equal quadrature fluctuation. A coherent state $|\alpha\rangle$, also called Glauber state, is defined as an eigenstate of the annihilation operator $\hat{a}$ with complex eigenvalue α [6, 12, 17]

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{1.6}$$

The solution of this eigenvalue problem is written as [6]

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = D(\alpha)|0\rangle, \tag{1.7}$$

where $|0\rangle$ is the vacuum state, $\hat{D}(\alpha)$ is the displacement operator defined by

$$\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}}. \tag{1.8}$$

This operator obeys the following relations [18, 19]

$$\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha, \tag{1.9a}$$
$$\hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{D}(\alpha) = \hat{a}^\dagger + \alpha^*. \tag{1.9b}$$

The expectation values of $\hat{X}_{1,2}$ and $\hat{X}_{1,2}^2$ in the coherent state $|\alpha\rangle$ are

$$\langle\hat{X}_1\rangle = \frac{1}{2}(\alpha^* + \alpha), \tag{1.10a}$$
$$\langle\hat{X}_2\rangle = \frac{i}{2}(\alpha^* - \alpha), \tag{1.10b}$$
$$\langle\hat{X}_1^2\rangle = \frac{1}{4}(\alpha^{*^2} + 2\alpha^*\alpha + \alpha^2 + 1), \tag{1.10c}$$
$$\langle\hat{X}_2^2\rangle = -\frac{1}{4}(\alpha^{*^2} - 2\alpha^*\alpha + \alpha^2 - 1), \tag{1.10d}$$

and the variances are,

$$(\Delta\hat{X}_1)^2 = \langle\hat{X}_1^2\rangle - \langle\hat{X}_1\rangle^2 = \frac{1}{4}, \tag{1.11a}$$
$$(\Delta\hat{X}_2)^2 = \langle\hat{X}_2^2\rangle - \langle\hat{X}_1\rangle^2 = \frac{1}{4}. \tag{1.11b}$$

Then

$$(\Delta\hat{X}_1)(\Delta\hat{X}_2) = \frac{1}{4}, \tag{1.12}$$

which means that the coherent state is a minimum uncertainty state with equal quadrature fluctuations.

1.4.3 Squeezed vacuum state

A squeezed state $|\alpha, \xi\rangle$ can be generated by first operating with the squeeze operator $\hat{S}(\xi)$ on the vacuum state followed by the displacement operator $\hat{D}(\alpha)$ [12, 16, 18, 19], where the single mode squeeze operator $\hat{S}(\xi)$ can be defined as follows:

$$\hat{S}(\xi) = e^{\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2})} \tag{1.13}$$

where

$$\xi = re^{i\theta}, \qquad 0 \leq r < \infty, \qquad 0 \leq \theta \leq 2\pi \tag{1.14}$$

the real numbers r and θ are called the squeeze factor and the squeeze angle of the squeezed state, respectively. The squeeze operator satisfies the following operator relations [18],

$$\hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi) = \hat{a}cosh(r) - \hat{a}^\dagger e^{i\theta} sinh(r), \tag{1.15a}$$

$$\hat{S}^\dagger(\xi)\hat{a}^\dagger\hat{S}(\xi) = \hat{a}^\dagger cosh(r) - \hat{a}e^{-i\theta} sinh(r). \tag{1.15b}$$

The squeezed vacuum (SV) state can be expressed as follows,

$$|\alpha, \xi\rangle = \hat{D}(\alpha)\hat{S}(\xi)|0\rangle. \tag{1.16}$$

A classical electromagnetic field consists of waves with well-defined amplitude and phase while in quantum treatment the field can be described by the two conjugate quadrature components $\hat{X}_{1,2}$. The product of the uncertainties in the two conjugate quadrature components satisfies the minimum uncertainty principle. A coherent state has an equal amount of uncertainty in the two variables and their product is the minimum uncertainty. The product cannot be minimized any more, it is possible to reduce fluctuation in one quadrature below the quantum limit but the other quadrature fluctuation must be increased above quantum limit to satisfy Heisenberg uncertainty principle. These states are called squeezed states.

The relations (1.9, 1.15) are used to calculate the expectation values of $\hat{X}_{1,2}$

and $\hat{X}^2_{1,2}$ in the squeezed state $|\alpha, \xi\rangle$

$$\langle \hat{X}_1 \rangle = \frac{1}{2}(\alpha^* + \alpha), \tag{1.17a}$$
$$\langle \hat{X}_2 \rangle = \frac{i}{2}(\alpha^* - a), \tag{1.17b}$$
$$\langle \hat{X}^2_1 \rangle = \frac{1}{4}\big((\alpha + \alpha^*)^2 + e^{2r} sin^2(\frac{\theta}{2}) + e^{-2r} cos^2(\frac{\theta}{2})\big), \tag{1.17c}$$
$$\langle \hat{X}^2_2 \rangle = \frac{1}{4}\big(- (\alpha - \alpha^*)^2 + e^{2r} cos^2(\frac{\theta}{2}) + e^{-2r} sin^2(\frac{\theta}{2})\big), \tag{1.17d}$$

then,

$$(\Delta \hat{X}_1)^2 = \frac{1}{4}\big(e^{2r} sin^2(\frac{\theta}{2}) + e^{-2r} cos^2(\frac{\theta}{2})\big), \tag{1.17e}$$
$$(\Delta \hat{X}_1)^2 = \frac{1}{4}\big(e^{2r} cos^2(\frac{\theta}{2}) + e^{-2r} sin^2(\frac{\theta}{2})\big), \tag{1.17f}$$

Rotated quadrature operators $\hat{Y}_{1,2}$ are defined as follows [19]:

$$\hat{Y}_1 = \hat{X}_1 cos\big(\frac{\theta}{2}\big) + \hat{X}_2 sin\big(\frac{\theta}{2}\big), \tag{1.18a}$$
$$\hat{Y}_2 = -\hat{X}_1 sin\big(\frac{\theta}{2}\big) + \hat{X}_2 cos\big(\frac{\theta}{2}\big), \tag{1.18b}$$

then,

$$(\Delta \hat{Y}_1)^2 = \frac{1}{4} e^{-2r}, \tag{1.19a}$$
$$(\Delta \hat{Y}_2)^2 = \frac{1}{4} e^{2r}, \tag{1.19b}$$

consequently,

$$(\Delta \hat{Y}_1)(\Delta \hat{Y}_2) = \frac{1}{4}. \tag{1.20}$$

A general class of minimum uncertainty states is known as squeezed states. A squeezed state may have less noise in one quadrature than a coherent state. To satisfy the minimum uncertainty state the noise in the other quadrature is greater than that of coherent state. The coherent state is a particular member of this general class of minimum uncertainty states with equal noise in both quadratures.