شبكة المعلومات الجامعية
التوثيق الإلكتروني والميكروفيلم

بسم الله الرحمن الرحيم

MONA MAGHRABY

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جامعة عين شمس

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قسم

نقسم بالله العظيم أن المادة التي تم توثيقها وتسجيلها
علي هذه الأقراص المدمجة قد أعدت دون أية تغيرات

يجب أن

تحفظ هذه الأقراص المدمجة بعيدا عن الغبار

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On the Solvability of a Class of Operator- Differential Equations of an Arbitrary Order

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كامل الشكر والتقدير إلى السادة الأساتذة الذين قاموا بالإشراف على هذه الرسالة وهم :

- **أ.د/ نشأت فريد محمد فتحي**

  أستاذ الرياضيات البحتة المتفرغ

  كلية العلوم - جامعة عين شمس

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- **د/ عبدالباسط اسماعيل أحمد**

  مدرس الرياضيات الهندسية

  كلية الهندسة – جامعة حلوان

ثم أشكر كل من ساعدنى بالقول والفعل أثناء قيامى بهذا العمل لإخراجه فى أحسن صورة وكذلك أشكر أمي وإخوتى وزوجتى.

كذلك أشكر جامعة عين شمس – كلية العلوم – قسم الرياضيات.

وأود أن أهدي هذه الرسالة إلى المرحوم أبى (رحمه الله) وإلى أمى أطال الله فى عمرها.

والله الموفق,,,

List of Publications

1- **Mohamed A. Labeeb, Abdel Baset I. Ahmed and Labib Rashed,** Conditional solvability of the boundary value problem of a self- adjoint operator-differential equations in a sobolev - type space, International Journal of Applied Mathematics, vol. 32, no. 3, pp. 469- 478, (2019).

2- **Nashat Fareid, Labib Rashed, Abdel Baset I. Ahmed and Mohamed A. Labeeb,** Solvability of initial-boundary value problem of a Multiple characteristic fifth-order operator-differential equation, Journal of the Egyptian Mathematical Society, vol. 2019, no. 27- 37, pp. 1-7, (2019).

3- **Nashat Fareid, Abdel Baset I. Ahmed and Mohamed A. Labeeb**, Sufficient conditions for regular solvability of an arbitrary order Operator-differential equation with initial-boundary conditions, Advances in Difference Equations, vol. 2020, no. 104, pp. 1-14, (2020).

Introduction

The interest of studying initial value problems for operator-differential equations is increasing since the theory of operator-differential equations was established. This is due to the fact that these equations allow searching at ordinary differential operators in addition to partial differential operators. A large number of researches on various aspects of the theory of operator-differential equations (see [51-52], [60], [81-83]). Recently in correlation with the study of problem arising in applications greatly increased the interest of study of differential equations in Banach and Hilbert spaces, especially in the Sobolev spaces. Note also that these studies are a natural development of the theory of operator-differential equations and is closely related to the theory of semi-groups like in the works of T. Kato, E. Hille, R. Phillips, S. G. Krein, S. Agmon, and L. Nirenberg and reflected in the monograph of K-J. Engel and R. Nagel [53], [58]. The operator-differential equations often lead to the study of polynomial operator pencils, these pencils are the parameters of the investigated equation. It should be noted that the fundamental theory of operator pencils originates from the work of M. V. Keldysh [54] and developed subsequently in the works of Dzh. E. Allahverdiev, M. G. Gasymov, I. Ts. Gohberg, A. G. Kostyuchenko, M. G. Krein, S. G. Krein, G. K. Langer, V. B. Lidskogo, A.S. Markus, V. I. Matsaev, S. S. Mirzoev, G.V. Radzievskogo A. A. Shkalikov, S. Ya. Yakubov and others. During the study of operator-differential equations arising questions about the completeness, minimality, and the basis of their elementary solutions which can be closely related to the investigations of the completeness, minimality and the basis of the system of eigenvalues and associated eigenvectors of the resolvent operator pencils.

In this thesis, we provide an initial-boundary value problem for a class of operator-differential equations of fifth-order with multiple and complicated characteristics. We developed the problem for an arbitrary-order to cover more applications in the future. At the same time we focus on the correct and unique solvability in weighted Sobolev type spaces with an exponential weight of operator-differential equations and spectral polynomial operator pencils (see [2], [5], [26], [29], [45-48], [52], [66-69], [71], [72], [74], [76], [23], [42], [67]).

Note that the differential equation whose characteristic equations have a valid high or valid multiple roots, are widely used in the simulation of mechanics and engineering, in the problems of magneto-hydrodynamics in problems filtration [27], in the dynamic problems of arches and rings [76] in aerodynamics problems [56]. We now discuss the results in detail, which is consist of the present introduction, three chapters, and the list of references.

In the first chapter

In Sobolev type spaces, we study the well-posed and uniqueness solvability for initial-boundary value problem of a fifth order operator-differential equations with multiple and complicated characteristic in the whole axis.

Let- H separable Hilbert space with the scaler-product $(x,y),\ x,y\in H$, $A-$ Positive definite self-adjoint operator, in H, and H_{γ} $(\gamma\geq 0)-$scale Hilbert spaces generated operator A, i.e.

$$H_{\gamma}=D(A^{\gamma}),(x,y)_{\gamma}=(A^{\gamma}x,A^{\gamma}y),x,y\in D\left(A^{\gamma}\right).$$

We introduce the following Hilbert spaces:

$$L_2\left(R_+;H\right)=\left\{f(x):\|f\|^2_{\left(R_+;H\right)}=\int_0^{+\infty}\|f(x)\|_H^2\,dx<+\infty\right\},$$

$$w_2^5(R_+;H)=\{u(x):\|u\|^2{}_{W_2^5(R_+;H)}=\int_0^{+\infty}\left(\left\|\frac{d^5u(x)}{dx^5}\right\|_H^2+\|A^5u(x)\|_H^2\right)dx<+\infty\}$$

Here and hereinafter, the derivatives are understood in the sense of the distribution theory [61]. We shall study the solvability of the following Initial Value Problems:

(i) Of a multiple characteristic fifth-order operator-differential equation in Sobolev-type

Space $W_2^5\left(R_+;H\right)$:

$$\left(-\frac{d^2}{dx^2}+A^2\right)\left(\frac{d}{dx}+A\right)^3 u(\mathrm{x})+\sum_{j=1}^{5} A_j \frac{d^{5-j}}{dx^{5-j}} u(\mathrm{x})=f(\mathrm{x}), \tag{1}$$

$$x \in R_+=[0,+\infty)$$

$$\frac{d^s u(0)}{dx^s}=0,\ s=\overline{0,3}. \tag{2}$$

(ii) Of a complicated characteristic fifth-order operator-differential equation in Sobolev-type space $W_2^5(R_+;H)$:

$$\prod_{k=1}^{5}\left(\frac{d}{dx}-\mu_k A\right)u(x)+\sum_{j=1}^{4} A_j \frac{d^{5-j}}{dx^{5-j}} u(x)=f(x), \tag{3}$$

$$x \in R_+=[0,+\infty)$$

$$\frac{d^s u(0)}{dx^s}=0,\ s=\overline{0,3}, \tag{4}$$

where $\mu_1=1, \mu_2=\mu_3=\mu_4=\mu_5=-1$ and $A_j;\ j=\overline{1,4}.$ are linear unbounded operators.

In the second chapter

For an arbitrary order operator-differential equation with the weight exponential $\frac{-\alpha x}{2}$, $\alpha\in(-\infty,+\infty)$ in the space $W_2^{n+m}(R_+;H)$, we obtain the sufficient conditions for the well-posedness of a regular solvability of the initial-boundary value problem:

$$\prod_{k=1}^{m}\left(-\frac{d}{dx}+r_k A\right)\prod_{k=1}^{n}\left(\frac{d}{dx}+r_k A\right)u(\mathrm{x})+\sum_{j=1}^{n+m} A_j \frac{d^{n+m-j}}{dx^{n+m-j}} u(\mathrm{x})=f(\mathrm{x}), \tag{5}$$

$$x \in R_+=[0,+\infty)$$

$$\frac{d^s u(0)}{dx^s}=0,\ s=\overline{0,\mathrm{n}+m-2}. \tag{6}$$

For the above problem, we introduce the following Hilbert space:

$$L_2(R_+;H)=\left\{u(x):\|u\|^2_{L_2(R_+;H)}=\int_0^{+\infty}\|u(x)\|_H^2\,dx<+\infty\right\},$$

$$W_2^{n+m}(R_+;H)$$
$$=\left\{u(x):\|u\|^2_{W_2^{n+m}(R_+;H)}=\int_0^{+\infty}\left(\left\|\frac{d^{n+m}u(x)}{dx^{n+m}}\right\|^2_H+\left\|A^{n+m}u(x)\right\|^2_H\right)dx<+\infty\right\}.$$

Where A is positive definite self-adjoint operator, A_j ($j=\overline{1,n+m}$) are linear, generally unbounded operators, $f(x)\in L_2(R_+;H)$, $u(x)\in W_2^{n+m}(R_+;H)$.

Moreover, in problem (5), (6) for any positive integer $n\geq 1$ and $m=1$, we get the main our problem:

$$\left(-\frac{d}{dx}+A\right)\left(\frac{d}{dx}+A\right)^n u(\mathrm{x})+\sum_{j=1}^{n+1}A_j\frac{d^{n-j+1}}{dx^{n-j+1}}u(\mathrm{x})=f(\mathrm{x}),\quad x\in R_+=[0,+\infty) \tag{7}$$

$$\frac{d^s u(0)}{dx^s}=0,\ s=\overline{0,\mathrm{n}-1}, \tag{8}$$

where A-positive definite self-adjoint operator, and A_j , $j=\overline{1,n+1}$ are linear, generally, unbounded operators. In order to avoid repetition, we shall introduce the following definitions and theorems in the general case (7), (8).

Definition 1. If there is a vector-function satisfying the equation (7) almost everywhere, then it will be called a regular solution of equation (7,8).

Definition 2. If there is any regular-solution of (7,8), the following inequality holds

$$\|u\|_{W_2^{n+m}(R_+;H)}\leq const\|f\|_{L_2(R;H)},$$

then equation (7,8) is called a regular solvable.

Problems of solvability on infinite intervals studied in detail for operator-differential equations of the second, third and fourth order equations with multiple and complicated characteristics should be noted, for example [64], [72]. But in our case, we study the solvability of initial-boundary value problems of fifth order operator-differential equations with multiple and complicated characteristics, and the main study on the solvability of a class of operator-differential equations of an arbitrary order. The solvability of the whole axis for operator-differential equations of fourth order with multiple characteristic studied in (see [43], [51], [52]).

Theorem 1. Let A- positive definite self-adjoint operator, $A_j A^{-j}$, $j=\overline{1,5}.$ operators are bounded in H and holds the following inequality

$$\sum_{j=1}^{5} b_j \left\| A_j A^{-j} \right\|_{H \to H} < 1.$$

Then the Initial value problem (1), (2) is regularly solvable.

Definition 3. If the function $u(x) \in W_2^{n+1}(R_+;H)$ satisfies the equation (7) almost everywhere in R_+, it will be called a regular solution of the equation of (7).

Definition 4. If for any $f(x) \in L_2(R_+;H)$, there is a regular-solution of (7) for which the boundary conditions (8) are carried out in the sense of

$$\lim_{x \to 0} \left\| A^{(n+1)-i-\frac{1}{2}} \frac{d^i u(\mathrm{x})}{dx^i} \right\|_H = 0, \quad i = \overline{0, \mathrm{n}-1},$$

and inequality

$$\|u\|_{W_2^{n+1}(R_+;H)} \le const \|f\|_{L_2(R_+;H)},$$

then we say that the problem (7), (8) regularly solvable.

Issues regular solvability boundary value problems for operator-differential equations of fourth order, the main part has multiple characteristics were investigated, e.g., in (see [15], [23]) (this can be attributed to the work [41] and is available in them some references). In this section, conditions for regular solvability are obtained of the problem (7), (8). In this connection it contains these conditions finding exact values of the norms of intermediate derivatives operators in space

$$W_2^{n+1}(R_+;H)$$

$$= \left\{ u(\mathrm{x}) : u(\mathrm{x}) \in W_2^{n+1}(R_+;H),\, u(0) = \frac{du(0)}{dx} = \frac{d^2u(0)}{dx^2} = \frac{d^3u(0)}{dx^3} = \ldots = \frac{d^{n-1}u(0)}{dx^{n-1}} = 0 \right\},$$

relative to the norm of the operator generated by the main part of the equation (7). On such a connection for the first time in work S. S. Mirzoev [62] (on the separate consideration of calculations of norms of operators of intermediate derivatives in [65]). To estimate norms, factorization method was used in study polynomial operator beams which depend on the actual parameter. Subsequently, these results were developed in [2],[17]. P_0 is denoted by an operator from space $W_2^{n+1}(R_+;H)$ to space $L_2(R_+;H)$ as follows:

$$P_0 u(x) \equiv \left(-\frac{d}{dx}+A\right)\left(\frac{d}{dx}+A\right)^n u(x),\ u(x)\in W_2^{n+1}(R_+;H).$$

Theorem 2. The operator P_0 is the isomorphism between the space $W_2^{n+1}(R_+;H)$ *and* $L_2(R_+;H)$.

Theorem 2 shows that the norm $\|P_0 u\|_{L_2(R_+;H)}$ is equivalent to the norm $\|u\|_{W_2^{n+1}(R_+;H)}$ in space $W_2^{n+1}(R_+;H)$. Therefore, according to the Theorem of intermediate - accurate derivatives [61] are finite numbers, $j=\overline{1,n}$.

$$n_j = \sup_{0\neq u\in W_2^{n+1}(R_+;H)} \frac{\left\|A^{n-j+1}\frac{d^j u}{dx^j}\right\|_{L_2(R_+;H)}}{\|P_0 u\|_{L_2(R_+;H)}}$$

which based on the technique of [12].

Theorem 3. Let A- positive definite self-adjoint operators, and $A_j A^{-j}$, $j=\overline{1,n}$, are bounded in H and the following inequality

$$\sum_{j=1}^{n} a_j \left\|A_j A^{-j}\right\|_{H\to H} <1, \qquad j=\overline{1,n},$$

holds. Where

$$a_j = \frac{1}{(n+1)^{\frac{n+1}{2}}}(j)^{\frac{j}{2}}(n-j+1)^{\frac{n-j+1}{2}}, \qquad j=\overline{1,n},$$

and

$$a_j = \sqrt{b_s} = \frac{1}{(n+1)^{n+1}} s^s (n-s+1)^{n-s+1} \quad , \quad s = \overline{1,n}.$$

Then the boundary value problem (7), (8) is regularly solvable. We subject the natural factorization -arising in the study operator pencil

$$P_j(\lambda;\beta;A) = \left((i\lambda)^2 E + A^2\right)^{n+1} - \beta(i\lambda)^{2j} A^{2(n-j+1)}, \; j = \overline{1,n} . \qquad (9)$$

Depending on a real parameter β, where E - the identity operator.

Theorem 4. Let $\beta \in \left[0, b_s^{-1}\right)$, $s = \overline{1,n}$, then polynomial operator Pencils (9) are invertible on the imaginary axis and they allow the following representations:

$$P_s(\lambda;\beta;A) = F_s(\lambda;\beta;A) F_s(-\lambda;\beta;A) \;, \quad s = \overline{1,n} \;.$$

Where

$$F_s(\lambda;\beta;A) = \prod_{r=1}^{n+1}\left(\lambda E - \omega_{s,r}(\beta) A\right) \equiv \sum_{m=0}^{n+1} \alpha_{m,s}(\beta) \lambda^{n-m+1} A^m \,,$$

$$F_s(-\lambda;\beta;A) = \prod_{r=1}^{n+1}\left(\lambda E + \omega_{s,r}(\beta) A\right) \equiv \sum_{\nu=0}^{n+1} \alpha_{\nu,s}(\beta) (-\lambda)^{n-\nu+1} A^\nu \,,$$

and $\mathrm{Re}\,\omega_{s,r}(\beta) < 0$, , $r = \overline{1,n+1}$, the numbers $\alpha_{m,s}(\beta)$, $m = \overline{0,n+1}$, $\alpha_{\nu,s}(\beta)$, $\nu = \overline{0,n+1}$, are positive and satisfying the following system of equations:

$$P_s(\lambda;\beta;A) = \left((i\lambda)^2 E + A^2\right)^{n+1} - \beta(i\lambda)^{2s} A^{2(n-s+1)}$$

$$= \sum_{m=0}^{n+1}\left[\sum_{\nu=0}^{n+1} \alpha_{m,s}(\beta) \alpha_{\nu,s}(\beta) \lambda^{n-m+1} (-\lambda)^{n-\nu+1} A^{m+\nu}\right].$$

At the end of chapters 1and 2 the results on the solvability of the problem (1), (2) and problem (7), (8) are illustrated as an example of initial boundary value problems for PDEs. Moreover, proved completeness of derivative of pencils built by eigenvectors and associated vectors polynomial operator pencil-iterated multiple and complicated higher order characteristic and the corresponding boundary value problem on half axis.

In the third chapter

We study the analytic properties of the resolvent polynomial operator pencils. Moreover, it is proved the completeness theorem of elementary solutions of the homogeneous differential