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شبكة المعلومات الجامعية

بسم اللّه الرحمن الرحيم

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التوثيق الالكتروني والميكروفيلم

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التوثيق الإلكتروني والميكروفيلم

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شبكة المعلومات الجامعية

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بالرسالة صفحات لم ترد بالأصل

SOME ASPECTS OF SINGULARITIES IN GENERAL RELATIVITY

A THESIS
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SUMMARY

One of the most important subjects in General Relativity is the singularities of space-times. Most exact solutions of the Einstein's field equations have singularities. Many space-time singularities are recognized as divergence of the curvature towards the boundaries of space-times as described by simple exact solutions, for example the Friedmann solution and Schwarzschild solution. These considerations do not guarantee that the occurrence of singularities in all the exact solutions are always associated with the divergence of curvature. For example, the components of the curvature tensor in the Taub-NUT space-time are bounded, but this space-time is still singular. Various exact solutions of the Einstein equations have been inspected in order to formulate a good definition of a singularity which can be used in a reasonably straightforward way on sorts of examples that commonly arise in General Relativity. Unfortunately, most of these definitions encounter some difficults. Finally, Clarke [31] in his book "the analysis of space-time singularities" gave an appropriate definition for a space-time to be singular. A space-time is singular if it contains an incomplete geodesic (or more general curve) γ such that there is no extension for which γ is extendible.
Singularities are not points of the physical space-time, but just imply that in a maximal space-time some inextendible non-spacelike curve end at a finit value of the (generalized) affine parameter.

Historically there have been several approaches to define the set of singular points by considering them as a boundary to provide explicit boundary constructions for space-times. The first approach is the so called geodesic-boundary (g-boundary), given by Geroch [41]. It represents the limit points of all incomplete geodesics. The second is the causal or c-boundary of Geroch, Kronheimer and Penrose [45].

The most elegant approach is more general than the above notion, it represents limit points of all incomplete curves. This construction was performed by Schmidt [98] in the so called bundle boundary (b-boundary).
The classification scheme proposed by Ellis and Schmidt [39] gives the nature of these singularities. This scheme uses the behaviour of the curvature tensor near the singularity to devide singularities into quasi-regular singularities, nonscalar curvature singularities, and scalar curvature singularities.

A difficult problem occurs when one tries to define a topological structure of space-time at a singular point. Geroch [41] made his construction by identifying various classes of incomplete time-like geodesics. He pointed out that space-time together with its g-boundary is a topological space. The problem in such a definition is that there are many possible boundaries with no compell of physical reason for choosing one instead of another, so it is not obvious how to define a topology at a singular point. It was a common surprise when the general construction of the b-boundary turned out that in both Friedmann and Schwarzschild space-times the boundary points are not Hausdorff separated from the corresponding space-time. The aim of this thesis is to cure these pathologies by defining an appropaite topology on the boundary of space-time.

This thesis contains five chapters :

In chapter 1, we give a mathematical formulation of General Relativity and we introduce some basic properties of manifolds.

In chapter 2, we demonstrate all attempts to define a singular space-time and the difficulties which are encountered there to give a reasonable definition of a singular space-time. We also discuss in more detail the constructions of the space-time boundaries in the sense of Geroch (g-boundary) and in the sense of Schmidt (b-boundary). The last section of this chapter is devided into two subsections. The first is concerned with the strong curvature singularities, and the second with the conformal singularities, where we proved that if γ is an incomplete null geodesic terminating at a conformal singularity of a strongly causal space-time, then γ can be reparametrized, in a neighbourhood of the conformal singularity, by an affine parameter to give a complete null geodesic. The definition of a physical and unphysical conformal singularity is given, and we have shown that the Friedmann singularity is a physical singularity.

Chapter 3 is devoted to the study of differential spaces, which are a natural generalization of differentiable manifolds. Some of the geometrical properties of these spaces are given. To treat space-time as a differential space rather than a differentiable manifold gives the possibility to cure the classical singularity problem, and to regard the singular boundaries of space-time as internal domains of a corresponding differential space.

In chapter 4, which is the core of this thesis, we give two possibilities to define an appropriate topology on the boundaries of space-times. These definitions explain the topology on the boundaries independently of their construction in contrast to Schmidt's b-boundary. The idea of the first attempt is based on the idea of the g-boundary, but it can be applied to more general situations with minor and obvious modifications. The second attempt is concerned with the behaviour of the Jacobi fields at the boundary of space-time by examining what happens between nearby geodesics at the boundary. A number of examples are given to illustrate these attempts.

In chapter 5, we treat two problems. The first one is the computation of the differential dimension of some well-known singularities by using a local theory of differential spaces. It is shown that the differential dimension of the interior Schwarzschild singularity is 4-dimensional, while in the case of the closed Friedmann singularity it is found to be 5-dimensional.
The second one is to glue together two Schwarzschild space-times in their singularities. We have done this in order to discuss the behaviour of radial null geodesics with zero angular momentum.

Chapter 1

INTRODUCTION

In this chapter we lay the foundation for a precise, mathematical formulation of General Relativity introducing some basic properties of manifolds and tensor fields. Basic manifold theory can be found in an eminently digestible form in Brickell and Clark [12], Bishop and Crittenden [5], and more advanced material is given in Kobayshi and Nomizu [64]. Manifold theory for the use in physics can be found in Martin [73]. Important references on the geometrical physics of space-time are Hawking and Ellis [52] and Beem and Ehrlich [3].

In sec. 1.2 we sumarize some structures on a manifold. Sec. 1.3 will deal with the theory of presheaves and sheaves (for more detail, see for example, Wells [110] and Bredon [11]). In sec. 1.4 Jacobi fields are discussed. Finally, in sec. 1.5 we shall give a brief review of the causal structure of space-time.

1.1 Differential Manifold Structure

Let X be a topological space with the Hausdorff property, that is any two different points of X possess disjoint neighbourhoods. A topological space X is called **n-dimensional topological manifold**, if

every point $p \in X$ has a neighbourhood homeomorphic to $\mathbb{R}^n$. The existence of a local homeomorphism between X and $\mathbb{R}^n$ should be understood in the following sense:
A pair (U,ϕ), where U is an open subset of X, and $\phi : U \longrightarrow E$ a homeomorphism onto an open subset $E \subset \mathbb{R}^n$ is said to be a (local) **chart** on the manifold X. By definition n is the dimension of the topological manifold. If p is a point of U then $\phi(p)$ is a point of $\mathbb{R}^n$, so $\phi(p)$ is an n-tuple of real numbers. Let the i^{th} coordinate of $\phi(p)$ be $x^i(p)$. Then we have $\phi(p) = (x^1(p), ..., x^n(p))$. Since ϕ is continuous, each x^i is a real-valued continuous function defined on U. Furthermore, if we have $x^i(p) = x^i(q)$ (i = 1,..., n) for two points p, q of U, then p = q, since ϕ is one to one. That is, the point of U is determined by the n-tuple of real numbers $(x^1(p), ..., x^n(p))$. $(x^1(p), ..., x^n(p))$ is called the system of **local coordinates** of the point p of U with respect to the chart (U, ϕ), and the n-tuple $(x^1, ..., x^n)$ of functions on U is called the local coordinate system on (U, ϕ).

An n-dimensional topological manifold M is covered by a family $\{U_\alpha\}$ of open sets U_α homeomorphic to open sets in $\mathbb{R}^n$. Let e_α be an open set of $\mathbb{R}^n$ homeomorphic to U_α, and let φ_α be a homeomorphism from U_α onto e_α. We call the collection of charts, $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ a **coordinate neighbourhood system** or an **atlas**. The U_α are named coordinate neighbourhoods.

Definition 1.1.1 *An atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ of an n-dimensional topological manifold M is called a* **coordinate neighbourhood system** *or* **atlas of class** C^r, *or a* C^r **atlas**, *if $\mathcal{A}$ has the following property: for any α and β such that $U_\alpha \cap U_\beta$ is non-empty, the maps*

$$\left.\begin{array}{l} \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta), \\ \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta) \end{array}\right\} \qquad (1.1.1)$$

from open subsets of $\mathbb{R}^n$ into open subsets of $\mathbb{R}^n$ are of class C^r.