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Contents

Summary

The study of dynamic equations on time scales goes back to its founder Stefan Hilger [28], in order to unify, extend and generalize ideas from discrete, quantum, and continuous calculus to arbitrary time scale calculus. A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers. When the time scale equals the set of real numbers, the obtained results yields results of ordinary differential equations, while when the time scale is the set of integers, the obtained results yields results of difference equations. The new theory of the so - called " dynamic equation" is not only unify the theories of differential equations and difference equations, but also extends these classical cases to the so - called q- difference equations (when $\mathbb{T} = q^{\mathbb{N}_0} := \{q^t : t \in \mathbb{N}_0, \ q > 1\}$ or $\mathbb{T} = q^{\overline{\mathbb{Z}}} = q^{\mathbb{Z}} \cup \{0\}$) which have important applications in quantum theory (see [31]).

In the last two decades, there has been increasing interest in obtaining sufficient conditions for oscillation (nonoscillation) of the solutions of dynamic equations on time scales. So we choose the title of the thesis " Oscillation Criteria for Some Dynamic Equations With Maxima on Time Scales".
This thesis is devoted to

1. Illustrate Hilger's theory by giving a general introduction to the theory of dynamic equations on time scales.

2. Summarize some of the recent developments in oscillation of first order delay differential equations and dynamic equations on time scales.

3. Establish some new sufficient conditions to ensure that all solutions of first order forced delay dynamic equations with maxima on time scales are oscillatory.

4. Give a comparison between the current results and the previous one. Latter, we give some examples to illustrate the importance of the presented results.

This thesis contains three chapters:

Chapter 1 is the introductory chapter and contains the basic concepts and some preliminary results of the oscillation theory of first order delay differential equations with maxima.

Chapter 2 consists of three sections. In the first section, we give an introduction to the theory of dynamic equations on time scales, differentiation, integrations, and some examples of time scales. Also, we present various properties of generalized exponential function on arbitrary time scale. In the second section, we establish some new oscillation results for the first order dynamic equations with maxima

$$x^{\Delta}(t) + q(t) \max_{s\in[t-\delta,t]} x(s) = 0,$$

$$x^{\Delta}(t) + p_1(t) \max_{s\in[\tau_1(t),t]} x(s) - p_2(t) \max_{s\in[\tau_2(t),t]} x(s) = 0,$$

and

$$x^{\Delta}(t) + \sum_{i=0}^{n} p_i(t) \max_{s\in[\delta_i(t),\sigma(t)]} x(s) + \sum_{j=1}^{m} q_j(t)x(\eta_j(t)) = 0.$$

The results of this section published in:
International Journal of Scientific and Innovative Mathematical Research, 5 (2017), 1-8.
In the third section, we establish some new oscillation criteria for first order forced delay dynamic equation of the form

$$x^{\Delta}(t) + \sum_{i=1}^{n} p_i(t) \max_{s\in[\tau_i(t),t]} x(s) + \sum_{j=1}^{m} q_j(t)x(\eta_j(t)) = q(t),$$

and

$$x^{\Delta}(t) + \sum_{i=1}^{n} p_i(t) \max_{s\in[t-\delta_i,t]} x(s) - \sum_{j=1}^{m} q_j(t) \max_{s\in[t-\eta_j,t]} x(s) = q(t).$$

The result of this section are published in:
International journal of Dynamical Systems and Differential Equations.

In Chapter 3, we establish some new oscillation criteria for first order sublinear delay dynamic equations with and without maxima of the form

$$x^{\Delta}(t) + p(t)x^{\alpha}(\tau(t)) = 0, t \geq t_0, \tag{1}$$

and

$$x^{\Delta}(t) + p(t) \max_{s \in [\tau(t), t]} x^{\alpha}(s) = 0, t \geq t_0, \tag{2}$$

where $p(t) \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$, α is a quotient of odd positive integer. Oscillation behavior of these equations is not studied before. In Section 3.1, we studied the oscillatory of first order superlinear delay dynamic equations with and without maxima on time scales and the obtained results was submitted for publication. Also, in section 3.2, we studied the oscillatory of first order sublinear delay dynamic equations with and without maxima on time scales and the obtained results was submitted for publication.

Chapter 1

Preliminaries

In this chapter, we give a survey of the previous studies related to the subject of the thesis and present some basic concepts of the theory of functional differential equations. Also, we sketch some preliminary results that can be used in this thesis and introduce some of the recent developments in oscillation theory of first order delay differential equations. Finally, we investigate the existence of oscillatory solutions of first order delay differential equations.

1.1 Introduction to delay-differential equations

Delay-differential equations (DDEs) form a large and important class of dynamical systems. They are often arise in either natural or technological control problems. In these systems, a control monitors the state of the system, and make an adjustment to the system based on its observations. Since, these adjustments can not be made instantaneously, a delay arises between the observation and the control action.

In mathematics, delay differential equations (DDEs) are considered as differential equations in which the derivative of the unknown function at a certain time is given in terms of it's values at previous times.

In order to solve a delay equation, we need to consider the earlier values of x at each time step. Therefore, we need to specify an initial function which gives the behavior of the system prior to time 0 (assuming that we start at $t = 0$). This function has to cover a period of time as long as the longest delay. For example, the initial function of the class of equations with a single delay,

$$x^{'}(t) = f(x(t), x(t-\tau)),$$

must be a function $x(t)$ defined on the interval $[-\tau, 0]$.

1.2 Application

Most of the differential equation models in population dynamics have been derived from the simple equation:

$$\frac{dx(t)}{dt} = x(t),$$

where $x(t)$ denotes the density of population (or biomass) of a single species at time t.
An equation with delays in production and destruction, assuming there is no immigration or emigration, can be written as:

$$\frac{dx(t)}{dt} = \text{birth rate} - \text{death rate}.$$

For instance, if we consider a population of adult flies, then the production or recruitment of adult flies at time t depends on the population of adult at time $t-\tau$, where τ is the time required for the larvae to become adult. If the birth and death rates are governed by density dependent factors, then we have

$$\frac{dx(t)}{dt} = a(x(t-\tau)) - b(x(t))$$

where the functions $a(.)$ and $b(.)$ denote density dependent production (recruitment) and destruction (death) rates respectively. The oscillation theory of delay differential equations has been extensively developed during the past few years. New applications which involve delay differential equations continue to arise with increasing frequency in the modeling of diverse phenomena in physics, biology, ecology, and physiology. We refer, for example, [26].

1.3 Oscillation and nonoscillation

In this section, we define the oscillation of solutions with or without delay. Let us now consider the solution x of the equation

$$x''(t) + p(t)x(t-\tau(t)) = 0, \tag{1.1}$$

which exists on a ray $[T_x, \infty]$ and satisfies $\sup\{|x(t)| : \ t \geq T\} > 0$ for every $T \geq T_x$. In other words, $|x(t)| \not\equiv 0$ on any infinite interval $[T, \infty)$. Such a solution is sometimes called a regular solution. We usually assume that $p(t) \geq 0$ or $p(t) \leq 0$ in (1.1), and then prove that $p(t) \not\equiv 0$ on any infinite interval $[T, \infty)$.

Definition 1.3.1 *[1] A nontrivial solution $x(t)$ is said to be oscillatory if it has infinite number of zeros, that is, there exists a sequence of zeros $\{t_n\}$ such that $x(t_n) = 0$ and $\lim_{n\to\infty} t_n = \infty$. Otherwise, x is said to be nonoscillatory.*
For nonoscillatory solutions there exists t_1 such that

$$x(t) \neq 0 \qquad \text{for all} \qquad t \geq t_1.$$

Definition 1.3.2 *[48] A nontrivial solution $x(t)$ is called an eventually positive solution if there is $T \geq 0$ such that $x(t) > 0$ for $t \geq T$.*

Definition 1.3.3 *The delay differential equation is called oscillatory if all its solutions are oscillatory.*

Example 1.3.1 *[22] The delay equation*

$$x'(t) + x(t - \tfrac{\pi}{2}) = 0,$$

has the oscillatory solution $x(t) = \sin t$ which is caused by the delay.

Example 1.3.2 *[1] Consider the equation*

$$x''(t) + \tfrac{1}{2}x'(t) - \tfrac{1}{2}x(t - \pi) = 0 \qquad \text{for} \qquad t \geq 0,$$

whose solution $x(t) = 1 - \sin t$ has a sequence of multiple zeros.

Example 1.3.3 *[1] Consider the equation*

$$x''(t) - x(-t) = 0. \tag{1.2}$$

This equation has the oscillatory solution $x_1(t) = \sin t$ and the nonoscillatory solution $x_2(t) = e^t + e^{-t}$. Then equation (1.2) is not oscillatory.

1.4 Oscillation of first order delay differential equations

1.4.1 Equations with constant delay

In this section we study the oscillation results that have been obtained for the delay differential equation

$$x'(t) + p(t)x(t-\tau) = 0, \qquad t \geq t_0, \tag{1.3}$$

where

$$p \in C([t_0, \infty), [0, \infty)) \quad \textit{and} \quad \tau \textit{ is a positive constant.} \tag{1.4}$$

Theorem 1.4.1 *[1] Assume that (1.4) holds and that there exists $\bar{t_0} > t_0 + \tau$ such that*

$$\int_{t-\tau}^{t} p(s)ds \geq \frac{1}{e} \quad for\ all \quad t \geq \bar{t_0} \tag{1.5}$$

and

$$\int_{t_0+\tau}^{\infty} p(t)[exp(\int_{t-\tau}^{t} p(s)ds - \frac{1}{e}) - 1]dt = \infty. \tag{1.6}$$

Then every solution of (1.3) is oscillatory.

Corollary 1.4.1 *[1] Assume that (1.4) holds. If*

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)ds > \frac{1}{e}, \tag{1.7}$$

then every solution of (1.3) is oscillatory.

Corollary 1.4.2 *[1] Let (1.4) and (1.5) hold. If*

$$\int_{t_0+\tau}^{\infty} p(t)(\int_{t-\tau}^{t} p(s)ds - \frac{1}{e})dt = \infty, \tag{1.8}$$

then every solution of (1.3) is oscillatory.

Example 1.4.1 *[1] Consider the delay differential equation*

$$x'(t) + (\frac{1}{1+t} + \frac{1}{e})x(t-1) = 0 \qquad for \qquad t \geq 0. \tag{1.9}$$

For $t \geq 1$*, we have*

$$\int_{t-1}^{t} (\tfrac{1}{1+t} + \tfrac{1}{e})dt = ln\tfrac{1+t}{t} + \tfrac{1}{e} > \tfrac{1}{e},$$

and

$$\lim_{t\to\infty} \int_{t-1}^{t} (\frac{1}{1+t} + \frac{1}{e})dt = \frac{1}{e}.$$

Hence (1.7) is not satisfied. Also for any $T > 1$*, we have*

$$\int_{1}^{T} (\tfrac{1}{1+t} + \tfrac{1}{e}) ln\tfrac{1+t}{t} dt \geq \tfrac{1}{e} \int_{1}^{T} ln\tfrac{1+t}{t} dt \to \infty \qquad as \qquad T \to \infty.$$

Therefore, by Corollary 1.4.2, every solution of (1.9) is oscillatory.

Theorem 1.4.2 *[1] Suppose that (1.4) holds,* $\int_{t}^{t+\tau} p(s)ds > 0$ *where* $t \geq t_0$ *for some* $t_0 > 0$*, and*

$$\int_{t_0}^{\infty} p(t) ln(e \int_{t}^{t+\tau} p(s)ds)dt = \infty. \tag{1.10}$$

Then every solution of (1.3) is oscillatory.

Theorem 1.4.3 *[39]. Assume that* p *and* τ *are positive numbers in (1.3). Furthermore, assume that* $p\tau e \leq 1$*. Then (1.3) has a nonoscillatory solution.*

Theorem 1.4.4 *[39]. Assume that* p *and* τ *are positive numbers in (1.3), then the necessary and sufficient condition for all solutions of (1.3) to be oscillatory is* $p\tau e > 1$*.*

Example 1.4.2 *[39]. The equation*

$$x'(t) + (\tfrac{1}{e})x(t-1) = 0,$$

has a nonoscillatory solution $x(t) = exp(-t)$*, by Theorem 1.4.3, because* $pe\tau = 1$*.*

1.4.2 Equations with real coefficients

In this section, we consider the delay differential equation

$$x'(t) + \sum_{i=1}^{n} p_i x(t - \tau_i) = 0, \tag{1.11}$$

where

$$p_i \in \mathbb{R} \quad \text{and} \quad \tau_i \in \mathbb{R}^+ \quad \text{for} \quad i = 1, 2, ..., n.$$

Theorem 1.4.5 *[26]. Assume that*

$$p_i\ \tau_i \geq 0 \quad \textit{for} \quad i = 1, 2, ..., n.$$

Then any of the following two conditions is sufficient for the oscillation of all solutions of (1.11).

1. $\sum_{i=1}^{n} p_i \tau_i > \frac{1}{e}$;
2. $(\prod_{i=1}^{n} p_i)^{1/n} (\sum_{i=1}^{n} \tau_i) > \frac{1}{e}$.

Corollary 1.4.3 *[26]. Consider the delay differential equation*

$$x'(t) + px(t - \tau) = 0, \tag{1.12}$$

where $p,\ \tau \in \mathbb{R}$. Then the following statements are equivalent.

1. *Every solution of (1.12) is oscillatory.*
2. $p\tau > 1/e$.

Now, consider equation (1.11) and assume that $p_{k_i} > 0$, $i = 1, 2, ..., l$ and that $p_{m_j} \leq 0$, $j = 1, 2, ..., r$ with $l + r = n$. If $q_{m_j} = -p_{m_j}$, $j = 1, 2, ..., r$, then the equation (1.11) takes the form

$$x'(t) + \sum_{i=1}^{l} p_i x(t - \tau_i) - \sum_{j=1}^{r} q_j x(t - \sigma_j) = 0, \tag{1.13}$$

where $p_i,\ \tau_i,\ q_j,$ *and* $\sigma_j \in \mathbb{R}^+$ for $i = 1, 2, ..., l,\ j = 1, 2, ..., r$ with $l + r = n$, $\tau_1 \geq \tau_2 \geq .... \geq \tau_l$ and $\sigma_1 \geq \sigma_2 \geq .... \geq \sigma_r$.