



بسم الله الرحمن الرحيم

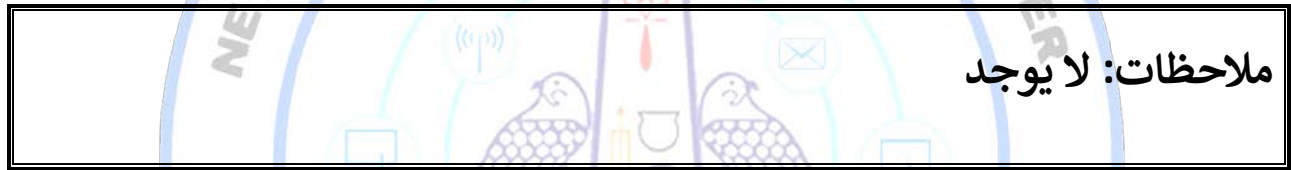
∞∞∞∞

تم رفع هذه الرسالة بواسطة / سامية زكى يوسف

بقسم التوثيق الإلكتروني بمركز الشبكات وتكنولوجيا المعلومات دون أدنى

مسئولية عن محتوى هذه الرسالة.

ملاحظات: لا يوجد





Faculty of Education
Department of Mathematics

Spectral Solutions of Differential Equations via Some New Classes of Orthogonal Polynomials and Special Functions

Presented by

Ahmed Gamal Atta Mohamed

A Thesis Submitted

to

Faculty of Education

In partial Fulfillment of the Requirements for
the Degree of Ph.D. in Teacher's Preparation of Science

(Pure Mathematics)

Department of Mathematics

Faculty of Education - Ain Shams University

Supervised by

Prof. Dr. Galal Mahrous Moatimid

Professor of Applied Mathematics
Department of Mathematics
Faculty of Education
Ain Shams University

Dr. Youssri Hassan Youssri

Associate Professor of Pure Mathematics
Department of Mathematics
Faculty of Science
Cairo University

2022

APPROVAL SHEET



Faculty of Education
Department of Mathematics

Candidate: Ahmed Gamal Atta Mohamed

Thesis Title: Spectral Solutions of Differential Equations via Some New Classes of Orthogonal Polynomials and Special Functions

Degree: Ph.D. in Teacher's Preparation of Science in Pure Mathematics
(Numerical Analysis)

Supervisors:

| No. | Name | Profession | Signature |
|-----|----------------------------------|--|-----------|
| 1. | Prof. Dr. Galal Mahrous Moatimid | Professor of Applied Mathematics Department of Mathematics Faculty of Education Ain Shams University | |
| 2. | Dr. Youssri Hassan Youssri | Associate Professor of Pure Mathematics Department of Mathematics Faculty of Science Cairo University | |

Addresses

Supervisor

1. **Prof. Dr. Galal Mahrous Moatimid**

Department of Mathematics
Faculty of Education
Ain-Shams University
E-mail: Gal_moa@edu.asu.edu.eg

2. **Dr. Youssri Hassan Youssri**

Department of Mathematics
Faculty of Science
Cairo University
E-mail: youssri@cu.edu.eg

Author

Ahmed Gamal Atta Mohamed

Department of Mathematics
Faculty of Education
Ain-Shams University
Email: ahmed_gamal@edu.asu.edu.eg

Abstract

In this thesis, we propose new efficient spectral techniques for handling certain types of partial differential equations and partial fractional differential equations such as; the one-dimensional linear hyperbolic partial differential equation of the first-order, the fractional diffusion wave equation, the heat conduction equation, the fractional initial-value problem, the time-fractional partial differential problem, the telegraph equation, and the nonlinear fractional Rayleigh-Stokes equation. In these techniques, we employ new basis functions of the shifted Chebyshev polynomials of the fifth and sixth kinds. The key idea of the presented techniques depends on transforming these equations with their underlying conditions into systems of algebraic equations in the unknown expansion coefficients. Our studies are supported by a careful convergence analysis of the suggested shifted fifth and sixth kinds Chebyshev expansions. Finally, some numerical examples are presented to confirm the accuracy and efficiency of the proposed techniques. We want to mention that this thesis consists of seven chapters and all the results of chapter two up to chapter seven are completely new. Some results obtained in this thesis are already published in four international prestigious journals with high impact factors and some others are submitted for publication and still under referring.

Contents

| | |
|---|------|
| Table of Contents | viii |
| List of Tables | x |
| List of Figures | xii |
| List of Abbreviations | xiii |
| Acknowledgements | xv |
| Summary | xvii |
| 1 Fundamentals | 1 |
| 1.1 Orthogonal polynomials | 5 |
| 1.1.1 A symmetric class of orthogonal polynomials | 6 |
| 1.1.2 Chebyshev polynomials of the fifth kind | 9 |
| 1.1.3 Chebyshev polynomials of the sixth kind | 11 |
| 1.2 Spectral methods | 13 |
| 1.2.1 Galerkin method | 14 |
| 1.2.2 Tau method | 15 |
| 1.2.3 Collocation method | 15 |
| 1.3 Fractional calculus | 15 |
| 1.3.1 Some definitions of the fractional calculus | 16 |
| 1.3.2 Properties of Caputo fractional differential operator | 17 |
| 1.3.3 Advantages of fractional derivatives | 19 |
| 1.3.4 Disadvantages of fractional derivatives | 20 |

| | | |
|----------|--|-----------|
| 1.3.5 | Some applications of fractional calculus | 20 |
| 2 | Shifted fifth-kind Galerkin treatment for linear hyperbolic | |
| | first-order partial differential equations | 23 |
| 2.1 | Introduction | 23 |
| 2.2 | Some properties and relations of Chebyshev polynomials of the | |
| | fifth-kind and their shifted ones | 25 |
| 2.3 | Shifted fifth-kind Chebyshev-Galerkin method for the one-dimensional | |
| | HPDEs of first-order | 28 |
| 2.3.1 | Choice of the basis functions | 28 |
| 2.3.2 | Numerical treatment of the one-dimensional HPDEs of | |
| | first-order | 32 |
| 2.4 | Convergence and error analysis | 35 |
| 2.5 | Illustrative examples | 41 |
| 2.6 | Conclusion | 48 |
| 3 | Shifted fifth-kind Chebyshev polynomials Galerkin based pro- | |
| | cedure for treating fractional diffusion-wave equation | 51 |
| 3.1 | Introduction | 51 |
| 3.2 | Some relations of the shifted fifth-kind Chebyshev polynomials | 53 |
| 3.3 | Galerkin approach for treatment of the FDWE | 56 |
| 3.3.1 | Selection of the basis functions | 57 |
| 3.3.2 | Some formulas concerned with the basis functions | 58 |
| 3.3.3 | Galerkin solution for FDWE | 62 |
| 3.3.4 | Transformation to the homogeneous initial and boundary | |
| | conditions | 65 |
| 3.4 | Galerkin approach for the treatment of FDWED | 67 |
| 3.4.1 | Galerkin solution for FDWED | 67 |
| 3.4.2 | Treatment of the non-homogeneous initial and boundary | |
| | conditions | 68 |
| 3.5 | Convergence and error analysis of the proposed double expansion | 68 |
| 3.6 | Illustrative examples | 71 |
| 3.7 | Concluding remarks | 75 |

| | | |
|----------|---|------------|
| 4 | Modal Shifted Fifth-Kind Chebyshev Tau Integral Approach | |
| | for Solving Heat Conduction Equation | 77 |
| 4.1 | Introduction | 77 |
| 4.2 | Some new formulas of fifth-kind Chebyshev polynomials and | |
| | their shifted ones | 79 |
| 4.2.1 | Derivation of the second-order derivatives formulas of $C_j(t)$ | 79 |
| 4.2.2 | Derivation of integrals formulas of $C_j(x)$ | 83 |
| 4.3 | A numerical tau approach for the treatment of heat conduction | |
| | equation | 86 |
| 4.3.1 | Treatment of the equation subject to homogeneous bound- | |
| | ary conditions | 87 |
| 4.3.2 | Handling heat conduction equation subject to the non- | |
| | homogeneous boundary conditions | 90 |
| 4.4 | Convergence and error analysis | 91 |
| 4.4.1 | Separable solution case | 91 |
| 4.4.2 | Non-separable solution case | 94 |
| 4.5 | Illustrative examples | 96 |
| 4.6 | Concluding remarks | 101 |
| 5 | A Novel Spectral Schemes to Fractional Problems with Non | |
| | Smooth Solutions | 103 |
| 5.1 | Introduction | 103 |
| 5.2 | Galerkin approach for treatment of FIVP | 105 |
| 5.2.1 | Basis functions | 105 |
| 5.2.2 | Galerkin solution for FIVP | 107 |
| 5.2.3 | Error bounds | 109 |
| 5.3 | Galerkin approach for treatment of time-fractional partial dif- | |
| | ferential problem | 112 |
| 5.3.1 | Basis functions | 112 |
| 5.3.2 | Galerkin solution for time-fractional partial differential | |
| | problem | 114 |
| 5.3.3 | Error bounds | 117 |
| 5.4 | Illustrative examples | 119 |

| | | |
|-------|---|-----|
| 5.5 | Concluding remarks | 124 |
| 6 | Advanced Shifted Sixth-Kind Chebyshev Tau Approach for Solving Linear One-Dimensional Hyperbolic Telegraph Type Problem | 125 |
| 6.1 | Introduction | 125 |
| 6.2 | Some relations of shifted Chebyshev polynomials of the sixth-kind | 128 |
| 6.3 | The proposed numerical scheme for treating the telegraph- type equation | 133 |
| 6.3.1 | Treatment of the non-homogeneous boundary conditions | 139 |
| 6.4 | Investigation of the convergence and error analysis | 140 |
| 6.5 | Illustrative examples | 144 |
| 6.6 | Concluding remarks | 150 |
| 7 | A fast Galerkin approach for solving the fractional Rayleigh-Stokes problem via sixth-kind Chebyshev polynomials | 151 |
| 7.1 | Introduction | 151 |
| 7.2 | Galerkin approach for treatment of the FRSE | 153 |
| 7.2.1 | Selection of the basis functions | 153 |
| 7.2.2 | Galerkin solution for the FRSE | 156 |
| 7.2.3 | Transformation to the homogeneous initial and boundary conditions | 158 |
| 7.3 | Convergence analysis | 158 |
| 7.4 | Illustrative examples | 162 |
| 7.5 | Concluding remarks | 166 |

List of Tables

| | |
|---|-----|
| 2.5.1 Comparison of the AE for Example 2.5.2 for $N = 6$. | 43 |
| 2.5.2 The MAE for Example 2.5.2 for different values of N and t . | 43 |
| 2.5.3 The AE for Example 2.5.2 for $N = 14$. | 44 |
| 2.5.4 The MAE for Example 2.5.2 for $N = 18$. | 44 |
| 2.5.5 Comparison of the best AE for Example 2.5.3 at $t = 0.1$ and | |
| $t = 0.5$. | 45 |
| 2.5.6 The AE for Example 2.5.3 for different values of N and t . | 46 |
| 2.5.7 MAE for Example 2.5.4 and CPU time (seconds) | 47 |
| 2.5.8 MAE for Example 2.5.5 and CPU time (seconds) | 48 |
| 3.6.1 The AE for Example 3.6.1. | 71 |
| 3.6.2 The MAE for Example 3.6.1 and CPU time (seconds). | 72 |
| 3.6.3 Comparison of the MAE of Example 3.6.2 for different values | |
| of α . | 73 |
| 3.6.4 Comparison of AE for Example 3.6.3. | 74 |
| 3.6.5 Comparison of the MAE of Example 3.6.3. | 74 |
| 3.6.6 The MAE for Example 3.6.3 and CPU time (seconds). | 74 |
| 4.5.1 The AE of Example 4.5.1. | 97 |
| 4.5.2 The AE of Example 4.5.1. | 97 |
| 4.5.3 MAE of Example 4.5.2. | 98 |
| 4.5.4 Comparison of the MAE of Example 4.5.2. | 98 |
| 4.5.5 The AE of Example 4.5.3. | 100 |
| 4.5.6 The AE of Example 4.5.3. | 100 |
| 5.4.1 Comparison of the MAE for Example 5.4.2. | 121 |

| | |
|---|-----|
| 5.4.2 Comparison of the MAE for Example 5.4.2. | 122 |
| 5.4.3 Comparison of the MAE for Example 5.4.2. | 123 |
| 5.4.4 The AE for Example 5.4.3. | 123 |
| 6.5.1 The AE for Example 6.5.1 at $M = 10, \ell = \tau = 1$ | 145 |
| 6.5.2 Comparison of MAE for Example 6.5.1 at $M = 8, \ell = \tau = 1$ | 145 |
| 6.5.3 The AE for Example 6.5.1 at $M = 13, \gamma = \delta = 1, \ell = 1, \tau = 2$. | 145 |
| 6.5.4 CPU time (seconds) of Example 6.5.1 | 145 |
| 6.5.5 CPU time (seconds) of Example 6.5.1 | 146 |
| 6.5.6 The AE for Example 6.5.1 at $M = 18, \ell = 1, \tau = 10$ | 146 |
| 6.5.7 Comparison of AE for Example 6.5.2 at $M = 10, \ell = \tau = 1$ | 147 |
| 6.5.8 The AE for Example 6.5.2 at $M = 14, \ell = 1, \tau = 2$ | 148 |
| 6.5.9 CPU time (seconds) of Example 6.5.2 | 148 |
| 6.5.10 The AE of Example 6.5.3 at $\ell = \tau = 1, \gamma = \delta = 1$ | 149 |
| 6.5.11 The AE for Example 6.5.3 at $M = 16, \ell = 1, \tau = 5, \gamma = 6, \delta = 9$ | 149 |
| 6.5.12 CPU time (seconds) of Example 6.5.3 | 150 |
| 7.4.1 The L_2 errors for Example 7.4.1. | 162 |
| 7.4.2 The L_∞ errors for Example 7.4.1. | 163 |
| 7.4.3 The L_2 errors for Example 7.4.2. | 164 |
| 7.4.4 The AE for Example 7.4.2. | 164 |
| 7.4.5 The AE for Example 7.4.3. | 165 |

List of Figures

| | |
|--|-----|
| 2.5.1 The MAE of Example 2.5.2 at $\ell = 1$ and different values of N . | 43 |
| 2.5.2 The exact and numerical solutions for Example 2.5.2 at $N = 14$. | 44 |
| 2.5.3 The graphs of the approximate solution (left side) and AE (right side) for Example 2.5.3 for $\ell = 1$ and $N = 10$. | 46 |
| 2.5.4 The exact and computed approximations for Example 2.5.3 at $N = 14$. | 46 |
| 2.5.5 The AE graphs of Example 2.5.4 at different values of t and for $N = 10$. | 47 |
| 2.5.6 The exact and approximate solutions at $N = 16$. for Example 2.5.5 | 48 |
| 3.6.1 The AE graphs of Example 3.6.1. | 72 |
| 3.6.2 The exact and approximate solutions for Example 3.6.2. | 73 |
| 3.6.3 The Log10(absolute error) of Example 3.6.3. | 74 |
| 4.5.1 The approximate solution and the MAE graphs of Example 4.5.1 | 97 |
| 4.5.2 The exact and approximate solutions of Example 4.5.2 | 98 |
| 4.5.3 MAE graphs of Example 4.5.2 | 99 |
| 4.5.4 The approximate solution and the MAE graphs of Example 4.5.3 | 100 |
| 5.4.1 The MAE for Example 5.4.1. | 119 |
| 5.4.2 The AE graph of Example 5.4.2. | 120 |
| 5.4.3 The AE graph of Example 5.4.2. | 121 |
| 5.4.4 The MAE graph of Example 5.4.2. | 123 |
| 5.4.5 The approximate and exact solutions of Example 5.4.3. | 124 |

| | | |
|---|-----------|-----|
| 6.5.1 The $\text{Log10}(AE)$ of Example 6.5.1 for different values of M . | . . | 146 |
| 6.5.2 The L_∞ Error of Example 6.5.1. | | 146 |
| 6.5.3 The $\text{Log10}(AE)$ of Example 6.5.2 for different values of M . | . . | 148 |
| 6.5.4 The $\text{Log10}(AE)$ of Example 6.5.3 for different values of M . | . . | 149 |
| 7.4.1 The L_∞ error for Example 7.4.1. | | 163 |
| 7.4.2 The L_∞ error for Example 7.4.2. | | 164 |
| 7.4.3 The L_∞ error for Example 7.4.3. | | 165 |

List of Abbreviations

| | |
|------------------|---|
| AE..... | Absolute error |
| MAE..... | Maximum absolute error |
| HPDEs..... | Hyperbolic first-order partial differential equations |
| FDWE..... | Fractional diffusion-wave equation |
| FDWED..... | Fractional diffusion wave equation with damping |
| FIVP..... | Fractional initial value problem |
| FRSE..... | Fractional Rayleigh-Stokes equation |
| CPU..... | Computational time |
| $C_i(x)$ | The shifted Chebyshev polynomials of the fifth-kind |
| $Y_i^*(x)$ | The shifted Chebyshev polynomials of the sixth-kind |

Acknowledgements

First of all, my gratitude and thanks to gracious **Allah** who always helps and guides me. I would like to thank **the prophet Mohamed** “peace be upon him” who urges us to seek knowledge and who is the teacher of mankind.

I would like also to thank the supervision committee:

Prof. Dr. Galal Mahrous Moatimid, Professor of Applied Mathematics, Faculty of Education , Ain Shams University, for accepting supervision of me, learning me ethics and scientific research assets, his good care for me and for his continuous support to reach the best.

Dr. Youssri Hassan Youssri, Associate Professor of Pure Mathematics, Faculty of Science, Cairo University, for suggesting the topic of the thesis, who provided me with guidance and continuous encouragement. He offered me much of his precious time and provided me with his wisdom and knowledge through many discussions we had.

Prof. Dr. Waleed Mohamed Abd-Elhameed, Professor of Pure Mathematics, Faculty of Science, Cairo University, who provided me with guidance and continuous encouragement. He did his best for the success of this work through precious comments, valuable reviews and remarks. His passion and extraordinary dedication to work have always inspired me and encouraged me to work harder.

Also, I would like to thank **Prof. Dr. Ehab Fathy Abd-Elfattah** and the head of Mathematics Department **Prof. Dr. Mohamed Yahia Abou-zeid** for providing me with all facilities required for the success of this work.