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A Study on Some Classes of Generalized Convex Functions

A Thesis

Submitted in Partial Fulfillment of the Requirements of the Doctor's Philosophy Degree in Teacher's Preparation in Science

(Mathematical Analysis)

Submitted to:

The Department of Mathematics, Faculty of Education,
Ain Shams University

By

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Contents

Introduction

The convexity of functions plays a central role in many various fields, such as economics, mechanics, biological system, optimization, and other areas of applied mathematics. Throughout this thesis, let I be a nonempty, connected, and bounded subset of $\mathbb{R}$. A real valued function $f(x)$ of a single real variable x defined on I is said to be **convex**, if for all u, $v \in I$ and $\lambda \in [0,1]$, one has the inequality:

$$f(\lambda u + (1-\lambda)v) \leq \lambda f(u) + (1-\lambda)f(v). \tag{1}$$

At the beginning of the 20^{th} century, many generalizations of convexity were extensively introduced and investigated in a number of ways by numerous authors in the past and present. One way to generalize the definition of a convex function is to relax the convexity condition 1(for a comprehensive review, see the monographs [41]).

As it is well known, the notion of the ordinary convexity can be expressed in terms of linear functions. An important direction for generalization of the classical convexity was to replace linear functions by another family of functions. For instance, Beckenbach and Bing [6], [7] generalized this situation by replacing the linear functions with a family of continuous functions such that for each pair of points $p_1(x_1, y_1)$ and $p_2(x_2, y_2)$ of the plane there exists exactly one member of the family with a graph joining these points.

More precisely, let $\{F(x)\}$ be a family of continuous functions $F(x)$ defined on a real interval I then, a function $f : I \to \mathbb{R}$ is said to be sub F-function, if for any $u, v \in I$ with $u < v$, there is a unique member of $\{F(x)\}$ satisfying

1. $F(u) = f(u)$ and $F(v) = f(v)$,

2. $f(x) \leq F(x)$ for all $x \in [u, v]$.

The sub F-functions possess various properties analogous to those of classical convex functions [6], [8], [7], [10]. For example, if $f : I \to \mathbb{R}$ is sub F-function, then for any $u, v \in I$, the inequality

$$f(x) \geq F(x)$$

holds outside the interval (u, v).

Of course mathematicians were able before 1937 to generalize the notion of convex functions [55], [56], [63]. In 1908, Phragmén and Lindelöf (see, [55]) dealed with family of trigonometric functions. More precisely, a function $f : I \to \mathbb{R}$ is said to be trigonometrically ρ-convex, if for any arbitrary closed subinterval $[u, v]$ of I such that $0 < \rho(v - u) < \pi$, the graph of $f(x)$ for $x \in [u, v]$ lies nowhere above the unique ρ- trigonometric function, determined by the equation:

$$M(x) = M(x; u, v, f) = A \cos \rho x + B \sin \rho x,$$

where A and B are chosen such that $M(u) = f(u)$, and $M(v) = f(v)$. Equivalently, if for all $x \in [u, v]$

$$f(x) \leq M(x) = \frac{f(u) \sin \rho(v - x) + f(v) \sin \rho(x - u)}{\sin \rho(v - u)}.$$

Full details can be found in classic books [3], [46], [57], [58] or in the monographs like [48].

In 2016, Ali [4] introduced the definition of hyperbolically p-convex functions in the sense of Beckenbach. For particular choices of the two-parameter family

$$F(x) = H(x) = A \cosh px + B \sinh px, \ p \in \mathbb{R} \setminus \{0\},$$

where A and B are chosen such that $H(u) = f(u)$ and $H(v) = f(v)$ which is known as sub H-function.
Actually, this class of functions has three names:

- Sub H-functions in the sense of Beckenbach (see [6]), which have been introduced and studied in 2016, by Ali see [4].
- Hyperbolically convex functions are have been suggested also by Ali in 2016 [4] as they are analogous to the notion of trigonometrically convex functions which have been considered by Phragmén and Lindelöf (see [55]).
- Hyperbolic p-convex functions considered by Dragomir in 2018 (see [22]).

Currently, we choose hyperbolic p-convex functions as a name for our class for two reasons

- **First**
  The value of p can be used to distinguish between hyperbolic p-convex functions and hyperbolic p-concave functions (see Example 1.12.3).
- **Second**
  To avoid ambiguity between hyperbolically convex functions in non-Euclidean geometry and our class.

In 2016, Ali [4] introduced sub E-convex functions by dealing with a family $\{E(x)\}$ of exponential functions

$$E(x) = A \exp Bx,$$

where A, B are arbitrary constants. More precisely, [4], [28] a positive function $f : I \to (0, \infty)$ is said to be a sub E-convex function on I, if for all $x \in [u, v] \subset I$,

$$f(x) \leq E(x)$$

where A and B are chosen such that $E(u) = f(u)$ and $E(v) = f(v)$.

Finally, in this thesis we study some properties of classes of generalized convex functions which analogous to those of classical convex functions. Furthermore, we established some new integral inequalities of Hermite-Hadamard and Hermite-Hadamard-Fejér types. Also, we introduced a class $BE[a, b]$ of functions representable as a difference of two sub E-convex functions.

Summary

The aim of this thesis is to
1- Discuss some classes of the generalized convex functions in the sense of Beckenbach.
2- Study the main characterizations of sub E-convex functions.
3- Extend some properties and integral inequalities (such as: Hermite-Hadamard, Hermite-Hadamard-Fej$\acute{e}$r, Ostrowski and Trapezoid, ...) which are known for ordinary convex functions.
4- Show that some results introduced by H$\ddot{u}$seyin Budak [11], in (2019), are incorrect.

The thesis consists of five chapters:

Chapter 1

This chapter is an introductory chapter. It contains definitions and basic concepts that are used throughout this thesis. It is regarded as a short survey of the basic needed material.

Chapter 2

The goal of this chapter is to obtain some new inequalities of Hermite-Hadamard and Hermite-Hadamard-Fej$\acute{e}$r type inequalities via fractional integrals for trigonometric ρ-convex functions. Furthermore, we use the Riemann-Liouville fractional integral to present recent re-

sults on fractional integral inequalities for trigonometric ρ-convex functions. Also, we show that some results introduced by H$\ddot{u}$seyin Budak [11], in (2019), are incorrect. Moreover, a counter example is given to confirm our claim.

The results of this chapter are:

- under submission for publication.

Chapter 3

The purpose of this chapter is to get upper and lower estimates for product of two hyperbolic p-convex functions, which is analogous to Hermite-Hadamard type inequalities for product of two hyperbolic p-convex functions.

The results of this chapter are:

- Accepted for publication in Italian Journal of Pure and Applied Mathematics on August 3rd, 2021.

- Presented in the 3rd International Conference for Mathematics and Its Applications, 2020.

Chapter 4

The aim of this chapter is to study the standard functional operations of sub E-convex functions. Furthermore, we introduce a class $BE[a,b]$ of functions representable as a difference of two sub E-convex functions.